RELATIONS & FUNCTIONS

KEY CONCEPTS& IMPORTANT FORMULAE

Objectives :1. To make the students familiar with higher mathematics
2. To understand and apply the knowledge of relation and function.

KEY CONCEPTS

(i).Domain, Co domain &Range of a relation
(ii).Types of relations
(iii).One-one, onto & inverse of a function
(iv).Composition of function
(v).Binary Operations

IMPORTANT DEFINATIONS

1. **RELATION:** A Relation R from a set X to set Y is a subset of X ×Y.
   A Relation R from a set X to set X is called a relation on X.

2. **FUNCTION: A relation $f: A \to B$ is called a function if f relates every element of A to unique element in B.**

   Remark: Difference between a relation & a function

   Every function is a relation but converse need not to be true.
   For example: Let A = {1, 2, 3, 4, 5}, B= {a, b, c, d}. Let R be the relation
   From A to B defined as :
   R={(1,a) , (1,c), (3,d),(5,d)}. Let $f: A \to B$ **be the relation defined as**
   f={(1,a) , (2,c), (3,d),(5,d),(4,a)}, here R is a relation but not a function.
   And f is a function.

3. **BINARY OPERATION**: If $A \neq \emptyset$ be any set then a function $*: A \times A \to A$ is called a binary operation on A.

TYPES OF RELATION:

1. **Reflexive:** If $A \neq \emptyset$, then a relation $R: A \to A$ **is called reflexive if** f relates every element of A to itself.

2. **Symmetric**: If $A \neq \emptyset$, then a relation $R: A \to A$ is called symmetric if (a,b) ϵ R implies (b,a) ϵ R ∀ a, b ϵ A

3. **Transitive** :If $A \neq \emptyset$, then a relation $R: A \to A$ is called transitive if (a,b) ϵ R and (b,c) ϵ R implies (a,c) ϵ R ∀ a, b, c ϵ A .

Equivalence Relation: A relation R on A is called equivalence relation if R is reflexive, symmetric and transitive.

TYPES OF FUNCTIONS:

1. **One-one function (injective):** If $A, B \neq \emptyset$, then a function $\boldsymbol{f: A \rightarrow B}$ **is called** one-one function if f maps (relates) distinct elements of A
to distinct elements of B.
If f(x)=f(y) implies x=y.

2. **Onto function(surjective):** If $A, B \neq \emptyset$, then a function $\boldsymbol{f: A \rightarrow B}$ **is called** onto function if for every element y∈B,there exists x∈A
Such that f(x) =y.

3. **Bijective function**: If a function is one-one and onto, then it is a bijective function.
Note:If $\boldsymbol{f: A \rightarrow B}$ is a bijective function then f is also called invertible function.
If $\boldsymbol{f: A \rightarrow B}$ is a bijective function then $\boldsymbol{g: B \rightarrow A}$ is called inverse of f if g(y) =x ∀ y ∈B

COMPOSITION OF FUNCTION:

If $f: A \rightarrow B \,\&\, g: B \rightarrow C$ be two functions then $gof: A \rightarrow C$ is a function defined by $gof(x) = g(f(x)) \forall x \in A$.
Note : If $f: A \rightarrow B \,\&\, g: C \rightarrow D$ be two functions then $gof: A \rightarrow D$ is defined by $gof(x) = g(f(x) \forall\ x \in A$ provided Range f is a subset of Domain g .

Note: If $f: A \rightarrow B \,\&\, g: B \rightarrow A$ **be two functions such that** $gof(x) = x = fog(x)$
Then f is invertible & $f^{-1} = g$

Binary Operation: If $A \neq \emptyset$ be any set then a function $*: A \times A \rightarrow A$ is called a binary operation on A.

Properties of Binary operations:

1. **A Binary operation $*: A \times A \rightarrow A$ is called commutative if $a * b = b * a$ ∀ a,b∈ A**
2. **A Binary operation $*: A \times A \rightarrow A$ is called associative if $(a * b) * c = a * (b * C)$∀ a, b∈ A.**
3. **If $*: A \times A \rightarrow A$ is a binary operation then e∈ A is called identity element if**

$$a * e = e * a = a \ \forall a \in A.$$

4. **If $*: A \times A \rightarrow A$ is a binary operation then b∈ A is called inverse of a∈ A if**

$$a * b = b * a = e$$

IMPORTANT BOARD QUESTIONS

SECTION A

1. If f(x) = x+ 7 and g(x) = x-7, x∈R find (fog) (7).

Sol.1.Here (fog) (x)=f(g(x))

=f(x-7)

=(x-7)+7=x

2 .Let * be a binary operation defined by a* b = 2a + b – 3 .Find 3*4.

Sol. Given a* b = 2a + b – 3

&3*4 = 6 + 4 -3 = 7

3.If A={1,2,3,4,5},write the relation aRb such that a+b=8,a,b∈A.

Sol.Here R={(3,5),(5,3),(4,4)}

4. Prove that the f: R→R defined as f(x) =2x is one-one.

Sol. Let x,y∈ *R* be such that f(x)=f(y),

2x=2y

x =y. Therefore f is one-one.

SECTION B

1. Show that the relation R in the set Z of integers given by

R={(a,b): 2 divides a-b }

Solution:

Reflexivity: Since a - a = 0 is divisible by 2 for every a ϵ *Z*

Therefore (a, a) ϵ R

Hence it is reflexive

Symmetric: Let (a , b) ϵ R , a-b is divisible by 2

Then b-a is also divisible by 2

i.e, (b, a) ϵ R

Hence R is symmetric

Transitive : Let (a , b) ϵ R and (b , c) ϵ R

Therefore, a – b = 2m and b – c = 2n, where m, n ϵ Z

Adding them a- b + b – c = 2 (m + n)

We get a – c = 2 (m + n), where m + n ϵ Z

Thus (a, c) ϵ R

Hence R is also transitive.

Thus R is an equivalence relation in Z

2. Show that the relation R in the set R of real numbers, defined as R = {(*a*, *b*) : $a \le b^2$ } is neither reflexive nor symmetric nor transitive.

Sol. Clearly, for a = $1/2$, aRa is false because, $\frac{1}{2} \le \frac{1}{4}$ is not true

Hence R is not reflexive.

Clearly (1,3) ϵ R {because 1< 9}
but (3,1) ∉ R {because 9≤ 1} is not true.

Hence R is not symmetric.

Further,(5,4)) ϵ R &(4,2)) ϵ R

but (5,2) ∉ R {because 5≤ 4} is not true. Therefore R is not transitive.

5. If the function f :[-1,1] → R defined by $f(x) = \frac{x}{x+2}$, Prove that f is one-one.
Also find the inverse of the function f:[-1,1] → Range of the f.

Sol.f: [−1, 1] → R is given as $f(x) = \frac{x}{x+2}$

Let $f(x) = f(y)$.

$$\Rightarrow \frac{x}{x+2} = \frac{y}{y+2}$$
$$\Rightarrow xy + 2x = xy + 2y$$
$$\Rightarrow 2x = 2y$$
$$\Rightarrow x = y$$

∴f is a one-one function.

It is clear that f: [−1, 1] → Range f is onto.

∴f: [−1, 1] → Range f is one-one and onto and therefore, the inverse of the function:

f: [−1, 1] → Range f exists.

Let g: Range f → [−1, 1] be the inverse of f.

Let y be an arbitrary element of range f.

Since f: [−1, 1] → Range f is onto, we have:

$y = f(x)$ for same $x \in [-1, 1]$

$$\Rightarrow y = \frac{x}{x+2}$$
$$\Rightarrow xy + 2y = x$$
$$\Rightarrow x(1-y) = 2y$$
$$\Rightarrow x = \frac{2y}{1-y},\ y \neq 1$$

Now, let us define g: Range f → [−1, 1] as

$$g(y) = \frac{2y}{1-y},\ y \neq 1.$$

$$\text{Now, } (gof)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1 - \frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

$$(fog)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y} + 2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$$

$\therefore f^{-1} = g$

$\Rightarrow f^{-1}(y) = \dfrac{2y}{1-y}, y \neq 1$

4. Show that the relation R on set A = $\{1,2,3,4,5\}$ given by R= R= {(a, b):|a-b| is even}is an equivalence relation.

Sol. R = $\{(1,1)(1,3)(1,5)(2,2)(2,4)(3,1)(3,3)(3,5)(4,2)(4,4)(5,1)(5,3)\}$

Reflexive-

(a,a) $\in$ R as $|a-a|$ =0 is even number for every a belonging to A

Symmetric—

Let (a,b) $\in$ R $\Rightarrow |a-b|$ is even $\Rightarrow |b-a|$ is even $\Rightarrow (b,a) \in$ R

Transitive Relation—

If (a,b) $\in$ R $\Rightarrow |a-b|$ is even $\Rightarrow$ a-b = $\pm$ 2n

If (b,c) $\in$ R$\Rightarrow |b-c|$ is even $\Rightarrow$ b-c =$\pm$ 2m

a-c = a-b +(b-c) = $\pm$ 2(m+n)

$|a-c|$ is even number $\Rightarrow$ (a,c) $\in$ R

Hence R is an equivalence relation

5. Consider $f: R_+ \to [4,\infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse $f^{-1}$of f given by $f^{-1}(y) = \sqrt{y-4}$, where R+ is the set of all non-negative real numbers.

Sol.f(x) = $x^2 + 4$

$\therefore y = x^2 + 4$

$\therefore x = \sqrt{y-4}$

Let us define a function g:$[4,\infty) \to R_+$ such that,

$\therefore g(y) = \sqrt{y-4}$,

Now gof(x) = $g[f(x)]$

= g($x^2 + 4$)

$= \sqrt{x^2 + 4 - 4}$

$= x$

Similarly we can show fog(y) = y

Hence f is invertible with $f^{-1} = g$

$\therefore \; f^{-1}(y) = \sqrt{y - 4}$

SECTION C

1.Let f: N $\rightarrow$ N defined as f(x)=$9x^2 + 6x - 5$ show that f : N $\rightarrow$ S where S is the range of f is Invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$

Sol.$\because$ $f(x) = 9x^2 + 6x - 5$

$\therefore \; y = 9x^2 + 6x - 5$

$\Rightarrow \; x = \dfrac{-1 + \sqrt{y + 6}}{3}$

Let us define a function g:S $\rightarrow$ N such that,

$\therefore \; g(y) = \dfrac{-1 + \sqrt{y + 6}}{3},$

Now gof(x) = $g[f(x)]$

$= g(9x^2 + 6x - 5)$

$\dfrac{-1 + \sqrt{9x^2 + 6x - 5 + 6}}{3}$

$= = \dfrac{-1 + 3x + 1}{3}$

$= x$

Similarly we can show fog(y) = y

Hence f is invertible with $f^{-1} = g$

$\therefore \; f^{-1}(x) = \dfrac{-1 + \sqrt{x + 6}}{3}$

Now $f^{-1}(43) = \dfrac{-1 + \sqrt{43 + 6}}{3} = 2$

And $f^{-1}(163) = \dfrac{-1 + \sqrt{163 + 6}}{3} = 4$

2. Let A = Q×Q. Let * be a binary operation on A defined by (a,b)*(c,d)= (ac , ad+b). Show that * is commutative & Associative.

Find: (i) the identity element of A (ii) the invertible element of A.

Sol. A = QxQ

And (a,b) * (c,d) = (ac,b+ad) ∀ (a,b),(c,d) ∈ S

(a,b)*(c,d) =(ac,b+ad)

(I) commutative:

(a,b)*(c,d) =(ac,b+ad)

(c,d)*(a,b)=(ca,d+cb)

E.g. (1,2)*(3,4)=(3,6)

(3,4)*(1,2)=(3,10)

* is not commutative.

Associative:

{[(a,b)*(c,d)]*(e,f)}=(ac,b+ad)*(e,f)

=(ace,b+ad+acf)

{(a,b)*[(c,d)*(e,f)]}=(a,b)*(ce,d+cf)

=(ace,b+ad+acf)

* is associative.

(ii) if (e,e') is identity

(a,b)*(e,e')=(a,b) = (e,e')*(a,b)

(ae,b+ae')=(a,b)=(ea,e'+eb)

(ae,b+ae')=(a,b)

ae=a &b+ae'=b

e=1&e'=0, if a is not equal to 0.

Now, (a,b)=(ea,e'+eb)

a=ea, b= e'+eb

e=0, e'=b

Identity doesn't exist.

HOTS

Q1 Let A = { xϵ R : $-1 \leq x \leq 1$ } = B .Show that f:A → B given by f(x) = $x|x|$ is bijection .

Sol : We have $f(x) = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

a) f is one one

1)Let x , y $\epsilon[0,1]$ be such that

f(x) = f(y)

$x^2 = y^2$

$(x-y)(x+y) = 0$

$x = y \ or \ x = -y \ (rejected)$

2)Letx,yϵ $(-\infty, 0)$ *be such that*

f(x) = f(y)

$-x^2 = -y^2$

(x-y)(x+y)=0

$x = y \ or \ x = -y \ (rejected)$.

Therefore f is one one .

b) f is onto :

For every y$\epsilon[0,1]$, there exists x $\epsilon[0,1]$ s.t f(x) = y ie x^2 = y .

Also for every y $\epsilon(-\infty, 0)$,there exists x $\epsilon(-\infty, 0)$ s.t f(x)= y ie x^2 = - y .

Therefore f is onto .Hence f is a bijective function.

Q 2 If f(x) = $\sqrt{x}$,$x \geq 0$ and g(x) = $x^2 - 1$ are two real functions , then find fog and gof .

Sol: Here f(x) = $\sqrt{x}$,$x \geq 0$ and g(x) = $x^2 - 1$.

Domain f = [0 ,∞) and Range f = [0 ,∞)

Domain g = R and Range g = [-1 ,∞)

Computation of gof :

Therefore gof exists and gof :$[0,\infty) \to R$

gof (x) =g(f(x)) = g($\sqrt{x}$) = $(\sqrt{x})^2 - 1$

Computation of gof : We observe that Range g = [-1 ,∞) is not subset of Domain f .

Therefore Domain fog = { xϵ R and g(x) $\epsilon[0,\infty)$ }

= { xϵ R and $x^2 - 1\epsilon[0,\infty)$ }

= { xϵ R and $x^2 - 1 \geq 0$}

= { xϵ R and $x \leq -1$, $x \geq 1$ }

Domain fog =$(-\infty,1) \cup [1,\infty)$ and fog(x) = f(g(x)) =f($x^2 - 1$) = $\sqrt{x^2 - 1}$.

Q3 Let g(x) = 1+ $x - [x]$ and f(x) =$\begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$, then for all x , find fog (x) .

Sol : fog(x) = f(g(x)) = f(g(x)) = f(1+ $x - [x]$) = f (1+{x}) = 1

Because $\{x\} = x - [x]$.

Also $0 \leq x - [x] < 1$ ie $0 \leq \{x\} < 1$

$1 \leq 1 + \{x\} < 2$

Fog (x) = f (1+{x}) =1 [{x} denotes partial part or decimal part]

Q4 Two functions f :R→R and g : R → R are defined as f(x) = $\begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

and g(x) = $\begin{cases} -1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$. Find gof (e) + fog (π).

Sol : Here gof (e) + fog (π) = g(f(e)) + f(g(e)

= g(1) + f(0)

= $-1 + 0$

= -1

$A * B = A \cup B$ for all $A, B \in P(X)$. Prove that $*$ is commutative and associative.

Find the identity element. Also show that $\emptyset \in P(X)$ is the only invertible element.

Sol : We know that $A \cup B = A \cup C$ and $(A \cup B) \cup C = A \cup (B \cup C)$

Therefore for any $A, B, C \in P(X)$, we have

$A \cup B = A \cup C$ and $(A \cup B) \cup C = A \cup (B \cup C)$

ie $A * B = B * A$ and $(A * B) * C = A * (B * C)$.

Thus $*$ is both commutative and associative.

Now $A \cup \emptyset = A = \emptyset \cup A$ for all $A \in P(X)$

ie $A * \emptyset = \emptyset * A$ for all $A \in P(X)$

So $\emptyset$ is the videntity element.

Let $A \in P(X)$ be the invertible element. Then there exists $S \in P(X)$ s.t

$A * S = \emptyset = S * A$ ie $A \cup S = \emptyset = S \cup A$.

$S = \emptyset = A$.

Hence $\emptyset$ is the only invrrtible element.