APPLICATIONS OF DERIVATIVE

SOME IMPORTANT FORMULAE/ KEYCONCEPTS

1. RATE OF CHANGE OF QUANTITIES

Whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left[\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

EXAMPLE :

QUESTION :The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference? **(CBSE 2011)**

SOLUTION :Let any instant of time t, the radius of circle = r

Then, circumference $C = 2\pi r$

Diff. Both sides w.r.t t, we get

$$\frac{dC}{dt} = \frac{d}{dt}(2\pi r) = 2\pi\frac{dr}{dt}$$

Here, $\frac{dr}{dt} = 0.7\ cm/s$

$$\therefore \frac{dC}{dt} = 2\pi \times 0.7 = 1.4\pi\ cm/s$$

2. INCREASING AND DECREASING FUNCTIONS:

Let I be an open interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1<x_2$ in I $\Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in I$.

(ii) strictly increasing on I if $x_1<x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$

(iii) decreasing on I if $x_1<x_2$ in I $\Rightarrow f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in I$.

(iv) strictly decreasing on I if $x_1< x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$

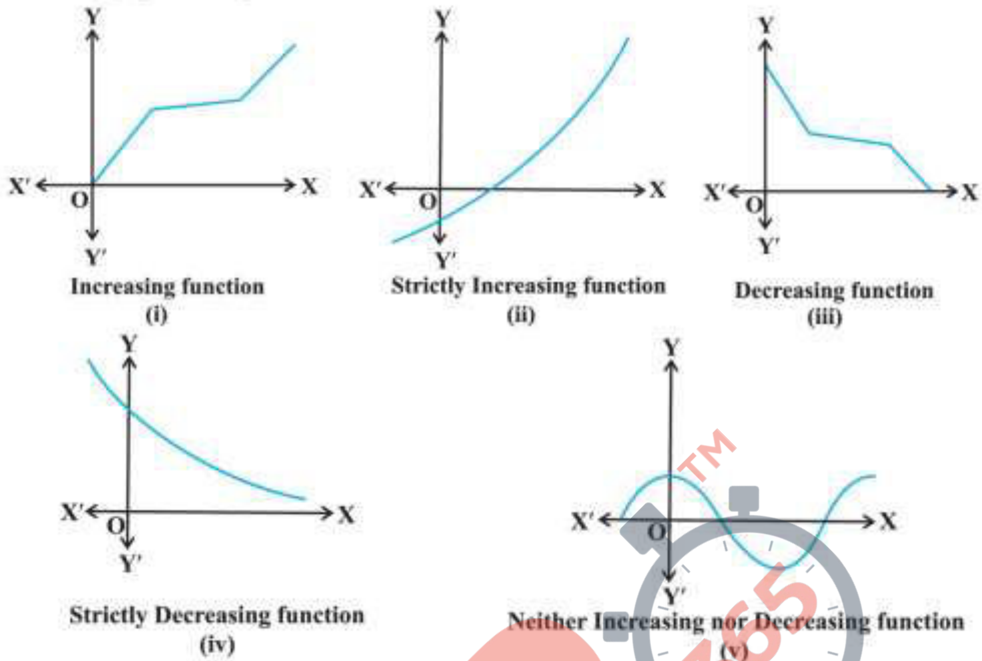
USING THE CONCEPTS OF DERIVATIVES

(i) f is strictly increasing in (a, b) if $f'(x) > 0$ for each $x \in (a, b)$

(ii) f is strictly decreasing in (a, b) if $f'(x) < 0$ for each $x \in (a, b)$

(iii) A function will be increasing (decreasing) in R if it is so in every interval of R.

For graphical representation



EXAMPLE

QUESTION : Find the interval in which the function $f(x)=2x^3 - 9x^2 +12x +15$ is (i) strictly increasing (ii) strictly decreasing

(CBSE2010)

SOLUTION: $f(x)=2x^3 - 9x^2 +12x +15$

$f'(x) = 6(x^2-3x+2)$

(i) For strictly increasing, $f'(x) > 0$

$6(x^2-3x+2) > 0$

$(x-1)(x-2) > 0$

$x< 1$ or $x > 2$

$x \in (-\infty, 1) \cup (2, \infty)$

so, f(x) is strictly increasing on $(-\infty, 1) \cup (2, \infty)$

(ii) for strictly decreasing $f'(x) < 0$

$6(x^2-3x+2) < 0$

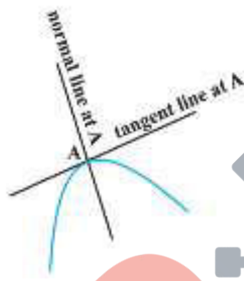
$(x-1)(x-2) < 0$

$1 < x < 2$

so, f(x) is strictly decreasing on (1,2).

3. TANGENTS AND NORMALS:

- Slope of the tangent to the curve $y = f(x)$ at the point (x_o, y_o) is given by $\left[\frac{dy}{dx}\right]_{(x_0, y_o)} (= f'(x_0))$.
- The equation of the tangent at (x_o, y_o) to the curve $y = f(x)$ is given by $y - y_o = f'(x_o)(x - x_o)$.



- Slope of the normal to the curve $y = f(x)$ at $(x_o, y_o) = -\frac{1}{\text{slope of the tangent at } (x_o, y_o)}$
- Slope of the normal to the curve $y = f(x)$ at (x_o, y_o) is $-\frac{1}{f'(x_o)}$.
- The equation of the normal (x_o, y_o) to the curve $y = f(x)$ is given by $y - y_o = -\frac{1}{f'(x_o)}(x - x_o)$.
- If slope of the tangent line is zero, then $\tan\theta = 0$ and so $\theta = 0$ which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point (x_o, y_o) is given by $y = y_o$.
- If $\theta \to \frac{\pi}{2}$, then $\tan\theta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to they-axis. In this case, the equation of the tangent at (x_o, y_o) is given by $x = x_o$.

EXAMPLE

QUESTION: Find the equation of normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

SOLUTION:

$$y = x^3 + 2x + 6$$

$$\frac{dy}{dx} = (3x^2 + 2)$$

Slope of the tangent at $(x_1, y_1) =$ $3x_1^2 + 2$

Slope of normal at point (x_1,y_1) $= \frac{-1}{3x_1^2+2}$

Slope of the given line is $\frac{-1}{14}$

According to the given condition,

Slope of normal = Slope of line

$$\frac{-1}{3x_1^2+2} = \frac{-1}{14}$$

$$x_1 = \pm 2$$

When $x_1= 2$ $y_1=18$

When $x_1=-2$ $y_1=-6$

Equation of normal at (2,18) is

$y-18=\frac{-1}{14}(x-2)$

$x+14y=254$ ---(1)

Equation of normal at (-2,-6) is $y+6=\frac{-1}{14}(x+2)$

$x+14y+86=0$

4. Approximations:

$$f(x+\Delta x) = f(x) + \Delta y$$
$$\approx f(x) + f'(x)\,\Delta x \qquad \text{(as } dx = \Delta x\text{)}$$

- Increment Δy in the function y = f(x) corresponding to increment Δ x in x is given by $\Delta y = \frac{dy}{dx}\Delta x$.
- Relative error in y $=\frac{\Delta y}{y}$.
- Percentage error in y $=\frac{\Delta y}{y} \times 100$.

EXAMPLE

QUESTION: Evaluate $\sqrt[4]{81.5}$ **(CBSE 2012)**

SOLUTION: :*Let x =81 and Δx =0.5*

$$y = \sqrt[4]{x} = \sqrt[4]{81} \ldots\ldots\ldots\ldots(1)$$

$$y + \Delta y = \sqrt[4]{81.5} \ldots\ldots\ldots(2)$$

$$\Delta y = \sqrt[4]{81.5} - \sqrt[4]{x}$$

$$\sqrt[4]{81.5} = \Delta y + \sqrt[4]{x}$$

using approximation $\Delta y \cong dy$

$$\sqrt[4]{81.5} = \frac{dy}{dx}.dx + \sqrt[4]{x}$$

$$= \frac{1}{4} x^{-\frac{3}{4}} .0.5 + 3$$

$$= \frac{1}{4} \times \frac{1}{81^{3/4}} \times \frac{1}{2} + 3$$

$$= \frac{1}{4} \times \frac{1}{27} \times \frac{1}{2} + 3 = \frac{1}{216} + 3 = 3.0046$$

5. MAXIMA AND MINIMA:

- Let f be a function defined on an interval I. Then
  (a) f is said to have a maximum value in I, if there exists a point c in I such that f (c) ≥ f (x) , for all x ∈ I.
- The number f (c) is called the maximum value of f in I and the point c is called a point of maximum value of f in I.
  (b) f is said to have a minimum value in I, if there exists a point c in I such that f (c) ≤ f (x), for all x ∈ I.
- The number f (c), in this case, is called the minimum value of f in I and the point c, in this case, is called a point of minimum value of f in I.

- f(x) is said to have an extreme value in I if there exists a point c in I such that f (c) is either a maximum value or a minimum value of f in I.
- The number f (c), in this case, is called an extreme value of f in I and the point c is called an extreme point.

Absolute maxima and minima

- Let f be a function defined on the interval I and $c \in$ I. Then
  (a) f(c) is absolute minimum if $f(x) \geq f(c)$ for all $x \in$ I.
  (b) f(c) is absolute maximum if $f(x) \leq f(c)$ for all $x \in$ I.

(c) $c \in$ I is called the critical point off if $f'(c) = 0$

(d) Absolute maximum or minimum value of a continuous function f on [a, b] occurs at a or b or at critical points off (i.e. at the points where f ′is zero)

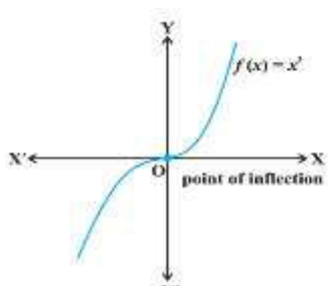
- If $c_1, c_2, \ldots, c_n$ are the critical points lying in [a , b], then
- absolute maximum value of f = max{f(a), f(c1), f(c2), … , f(cn), f(b)}
  and absolute minimum value of f = min{f(a), f(c1), f(c2), … , f(cn), f(b)}.

Local maxima and minima

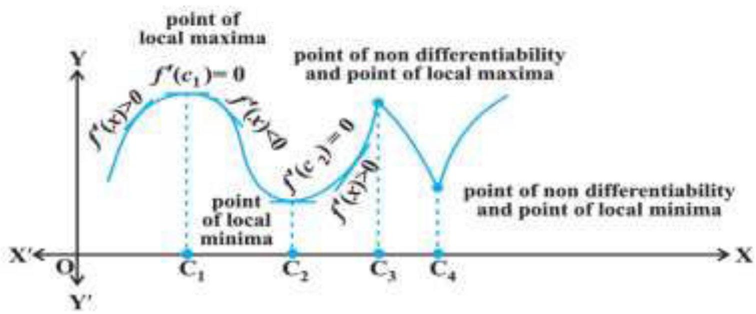
- (a)A function f is said to have a local maxima or simply a maximum vaJue at x a if $f(a \pm h) \leq f(a)$ for sufficiently small h
- (b)A function f is said to have a local minima or simply a minimum value at x = a if $f(a \pm h) \geq f(a)$.
- ** First derivative test :A function f has a maximum at a point x = a if
  (i) $f'(a) = 0$, and
  (ii) $f'(x)$ changes sign from + ve to –ve in the neighborhood of 'a' (points taken from left to right).

However, f has a minimum at x = a, if
(i) $f'(a) = 0$, and
(ii) $f'(x)$ changes sign from –ve to +ve in the neighborhood of 'a'.

If $f'(a) = 0$ and $f'(x)$ does not change sign, then f(x) has neither maximum nor minimum and the point 'a' is called **point of inflection.**



- The points where $f'(x) = 0$ are called stationary or critical points. The stationary points at which the function attains either maximum or minimum values are called extreme points.



Second derivative test

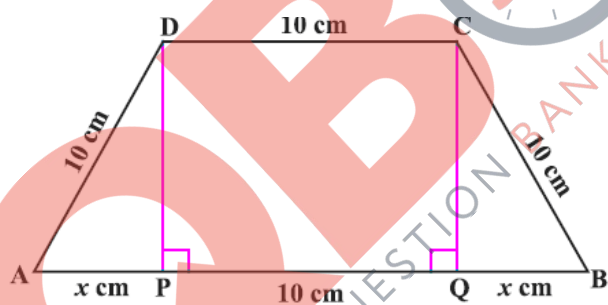
(i) a function has a maxima at x= a if f ′(x) =0 and f ″ (a) <0

(ii) a function has a minima at x = a if f ′(x) = 0 and f ″(a) > 0.

EXAMPLE

QUESTION: If length of three sides of a trapezium other than base is equal to 10cm each, then find the area of the trapezium when it is maximum.

SOLUTION: The required trapezium is as given in Fig below. Draw perpendiculars DP and CQ on AB



Δ APD is congruent to Δ BQC

Let AP = BQ= x cm

$$DP = QC = \sqrt{100 - x^2}$$

$$A.area\ of\ trapezium = (x+10)\sqrt{100 - x^2}$$

$$A'(x) = \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}$$

$$A'(x) = 0 \Rightarrow x = 5$$

$$A''(x) = \frac{2x^3 - 300x - 100}{\left(100 - x^2\right)^{\frac{3}{2}}}$$

$$A''(x)\Big|_{x=5} = \frac{-30}{\sqrt{75}} < 0$$

Thus the area of the trapezium is maximum at x=5

Area= $75\sqrt{3}$ cm^2

IMPORTANT BOARD QUESTIONS

4 MARK QUESTIONS

1. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x. **(CBSE 2008)**

SOLUTION: D = $\frac{3}{2}(2x+1)$

$$r = \frac{3}{4}(2x+1)$$

Volume of spherical balloon = $\frac{4}{3}\pi r^3$

V = $\frac{4}{3}\pi\left[\frac{3}{4}(2x+1)\right]^3$

$$\frac{dV}{dx} = \frac{9\pi}{16}[3(2x+1)^2].2$$

$$\frac{dV}{dx} = \frac{27\pi}{8}(2x+1)^2$$

2. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference? **(CBSE 2011)**

SOLUTION : Let any instant of time t, the radius of circle = r

Then, circumference C = $2\pi r$

Diff. Both sides w.r.t t, we get

$$\frac{dC}{dt} = \frac{d}{dt}(2\pi r) = 2\pi\frac{dr}{dt}$$

Here, $\frac{dr}{dt} = 0.7\ cm/s$

$\therefore \frac{dC}{dt} = 2\pi \times 0.7 = 1.4\pi\ cm/s$

3Find the interval in which the function f(x) = $(x+1)^3 (x-1)^3$ is

(i) strictly increasing **(ii) strictly decreasing** **(CBSE2010)**

SOLUTION: $f(x) = (x+1)^3(x-1)^3$

$f'(x) = 6(x+1)^2(x-3)^2(x-1)$

(i) For strictly increasing $f'(x) > 0$

$6(x+1)^2(x-3)^2(x-1) > 0$

$x-1 > 0$

$x > 1$

$x \in (1, \infty)$

so, f(x) is increasing on $(1, \infty)$

(ii) for strictly decreasing $f'(x) < 0$

$6(x+1)^2(x-3)^2(x-1) < 0$

$(x-1) < 0$

$x < 1$

$x \in (-\infty, 1)$

so, f(x) is decreasing on $(-\infty, 1)$

4. Find the interval in which the function f (x) =sinx +cosx, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing (CBSE 2009)

SOLUTION: f (x) =sinx +cosx

$f'(x) = \cos x - \sin x$

$f'(x) = \sqrt{2} \sin\left(\frac{\pi}{4} - x\right)$

(i) For strictly increasingf'(x) > 0

$$\sqrt{2}\sin\left(\frac{\pi}{4}-x\right) > 0$$

$$-\sqrt{2}\sin\left(x-\frac{\pi}{4}\right) < 0$$

$$\pi < x-\frac{\pi}{4} < 2\pi$$

$$\frac{5\pi}{4} < x < \frac{9\pi}{4}$$

$$\frac{5\pi}{4} < x < 2\pi$$

so, f(x) is strictly increasing on $(0,\frac{\pi}{4}) \cup \left(\frac{5\pi}{4},2\pi\right)$

(ii) for strictly decreasing f'(x) < 0

$$\sqrt{2}\sin\left(\frac{\pi}{4}-x\right) < 0$$

$$-\sqrt{2}\sin\left(x-\frac{\pi}{4}\right) > 0$$

$$0 < x-\frac{\pi}{4} < \pi$$

$$\frac{\pi}{4} < x < \frac{5\pi}{4}$$

so, f(x) is increasing on $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

5. Find the equations of all lines having slope 0, which are tangent to the curve $y=\frac{1}{x^2-2x+3}$

SOLUTION: $y=\frac{1}{x^2-2x+3}$

$$\frac{dy}{dx}=\frac{-(x-1)}{(x^2-2x+3)^2}=0$$

X=1

$$y=\frac{1}{2}$$

Equation of tangent

y-1/2=0

y=1/2

6. . Find the equation of tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_0,y_0)

SOLUTION:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_0y_0)}=\frac{b^2x_0}{a^2y_0}=m$$

$$y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$$

$$a^2yy_0-a^2y_0^2=b^2xx_0-b^2x_0^2$$

Equation of tangent

$$\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=-\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$$

Equation of normal

$$y-y_0=\frac{-1}{\frac{b^2x_0}{a^2y_0}}(x-x_0)$$

Equation of Normal

$$\frac{y-y_0}{a^2y_0}+\frac{x-x_0}{b^2x_0}=0$$

7. Find the approximate value of f(5.001)where $f(x)=x^3-7x^2+15$

SOLUTION:

$y + \Delta y = f(5.001)$……………………….....(1)

$x = 5 \quad \Delta x = 0.001$

$y = f(x)$……………………………………….(2)

(1) –(2) $\Rightarrow \Delta y = f(5.001) - f(x)$

$f(5.001) = \Delta y + f(x)$

$$= \frac{dy}{dx}.dx + f(5)$$

$= (3x^2 - 14x).\Delta x - 35$

$= (3.5^2 - 70).0.001 - 35$

$= 0.005 - 35$

$= -34.995$

8. If the radius of sphere is measured as 7 mtr with error of 0.02 m, than find the approximate error in calculating its volume.

SOLUTION:

$r = 7m,\ dr = 0.02m$

$V = 4/3\ \pi r^3$

$$dV = \frac{dv}{dr}.dr = \frac{4}{3}\pi 3r^2 dr = 4\pi 7^2 0.02$$

$dV = 3.92\pi\ m^3$

9. The length x of a rectangle is decreasing at the rate of 5 cm/min and the width y is increasing at the rate of 4 cm/min. When x=8 cm and y=6 cm, find the rates of change of (i) Perimeter (ii) area of the rectangle. (CBSE 2009)

SOLUTION:

Let at any instant of time t,

Perimeter, P = 2 (x +y) and Area, A = xy

Given, $\frac{dx}{dt} = -5\ cm/min$ and $\frac{dy}{dt} = 4\ cm/min$

(i) P = 2 (x + y)

$$\Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5+4) = -2\text{ cm / min}$$

Therefore, Perimeter of rectangle is decreasing at the rate of 2 cm/min

(ii)Here, A = xy , Diff. w.r.t t , we get

$$\Rightarrow \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = 8.(4) + 6.(-5) = 2\text{ cm}^2\text{ / min}$$

Therefore Area of the rectangle is increasing at the rate of 2 cm^2/min

10. A ladder 5m long is leaning against a wall. Bottom of ladder is pulled along the ground away from wall at the rate of 2m/s. How fast is the height on the wall decreasing when the foot of ladder is 4m away from the wall? (CBSE 2012)

SOLUTION:

Let AC is a ladder, $\frac{dx}{dt}$ = 2m/s so for $\frac{dy}{dt}$ at x=4m

$(AB)^2 + (BC)^2 = (AC)^2$

$x^2 + y^2 = 25$, x=4 then y = 3

diff. w.r.t. t both sides

We get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ so $4x2 + 3x\frac{dy}{dt} = 0$, $\frac{dy}{dt} = \frac{-8}{3}$m/s,

Hence height on the wall is decreasing at the rate of $\frac{8}{3}$ m/s

6 MARK QUESTIONS

1 Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

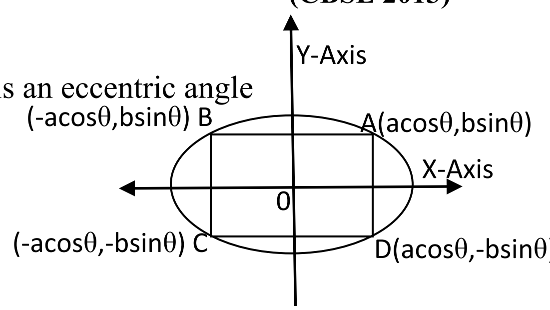
(CBSE 2013)

SOLUTION:

Let (acosθ, bsinθ) parametric coordinate of ellipse where θ is an eccentric angle

Length of AB = 2acosθ

Length of AD =2bsinθ



Area of rectangle ABCD = 2acosθ. 2bsinθ

=2absin2θ

Area of rectangle is greatest when sin2θ is greatest

sin2θ =1

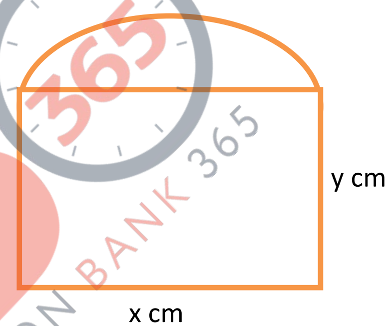
so $2\theta = 90^0$

$\theta = 45^{0}$

Area of greatest rectangle = 2ab when $\theta = 45^0$

2. A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum sunlight through the whole opening. Explain the importance of sunlight. (CBSE2011)

SOLUTION:



Let length of rectangle =2 x, Let breadth of rectangle =y

$2x + 2y + \pi x = 10$

$Y = (10 - 2x - \pi x)/2$

Finding area A, dA/dx, d^2A/dx

Put dA/dx = 0 we get $x = 10/(\pi+4)$

$d^2A/dx = -(\pi+4) < 0$ maximum area

$y = 10/(\pi+4)$

Comments on importance of sunlight

3. Find the point on the curve y2= 2x which is at minimum distance from the point (1,4) (CBSE 2009, 2011)

SOLUTION:

Let P(x,y) be any point $y^2 = 2x$ and M(1,4)

$$PM = \sqrt{(x-1)^2 + (y-4)^2}$$

$$= \sqrt{x^2 + y^2 - 2x - 8y + 17}$$

$$= \sqrt{x^2 - 8y + 17}$$

$$= \sqrt{\left(\frac{y^2}{2}\right)^2 - 8y + 17}$$

Let z = PM^2

$$z = \left(\frac{y^2}{2}\right)^2 - 8y + 17$$

$$\frac{dz}{dy} = y^3 - 8 \quad \text{and} \quad \frac{d^2z}{dy^2} = 3y^2$$

For maxima or minima

$$\frac{dz}{dy} = 0 \Rightarrow y = 2$$

$$\frac{d^2z}{dy^2} = 12 > 0\text{, it is minimum}$$

$Y^2 = 2x \Rightarrow x = 2$

PM is least when point is (2,2)

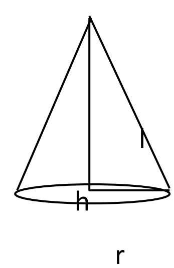
4. Show that the semivertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.(CBSE 2008,2011)

SOLUTION:

Let V be the Volume of the cone, then

$V = \frac{1}{3}\pi r^2 h \Rightarrow v^2 = \frac{\pi^2}{9}(r^4 l^2 - r^6)$

For max.or min $\frac{dv^2}{dr} = o \Rightarrow 2r^3(2l^2 - 3r^2) = o$



- ⇨ $2l^2 = 3r^2$
- ⇨ $l^2 = 3r^2/2$

or, $h^2 = \frac{r^2}{2}$

r= radius ,h= height l =slant height

Also, $\frac{dv^2}{dr^2} < o$ *at* $h = \frac{r^2}{2}$

α=semi vertical angle

Now $\frac{r}{h} = \tan\alpha$ *therefore* $\alpha = \tan^{-1}\sqrt{2}$

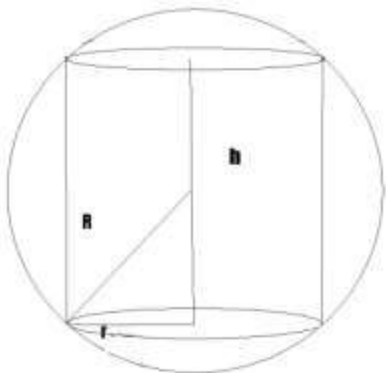
5. Show that the height of cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. (CBSE 2012, 2013)

SOLUTION:

Let h be the height of the cylinderhaving radii us rand R be the radius of the sphere.

Let V be the volume of the cylinder.

So, $V = \pi r^2 h = \pi(R^2 - \frac{h^2}{4})h = \pi R^2 h - \frac{\pi}{4}h^3$



For max. volume, $\frac{dV}{dh} = 0$

$\pi(R^2 - \frac{3h^2}{4}) = 0$

$h = \frac{2R}{\sqrt{3}}$

Also, $\frac{d^2V}{dh^2} < 0$ at $h = \frac{2R}{\sqrt{3}}$

There fore the required height $= \frac{2R}{\sqrt{3}}$

HOTS

1. Prove that : $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function in $\left[0, \frac{\pi}{2}\right]$.

   *(**HINT:** Find $\frac{dy}{dx}$ and show that it is >0)*

2. Prove that the curves $x = y^2$ and $xy = k$ are orthogonal if $8k^2 = 1$

   *(**HINT:** If the curves are Orthogonal , the tangents at point of intersection to the given curves are perpendicular i.e the product of slopes of the tangents = -1)*

3. Prove that the volume of the largest cone that can be inscribed in a sphere of radius a is $\frac{8}{27}$ of the volume of the sphere.

4. Find the sub intervals of $\left[0, \frac{\pi}{2}\right]$ in which the function $f(x) = sin^4 x + cos^4 x$ is
   (i) strictly increasing (ii) strictly decreasing.
5. Find the maximum area of the isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis
6. Show that the semi-vertical angle of the right circular cone of given total surface area and maximum volume is $\sin^{-1}\frac{1}{3}$.
7. Show that the volume of thegreatestcylinder that can be inscribed in acone of height h and semi vertical angle α is $\frac{4}{27}\pi h^3 tan^2 \alpha$.
8. Find the point on the curve $y = \frac{x}{1+x^2}$, where the tangent to the curve has the greatest slope.
9. Find the value of p for which the curves $x^2 = 9p(9 - y)$ and $x^2 = p(y + 1)$ cut each other at right angle.
10. Find the equation of the tangent to the curve

$y = \frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis