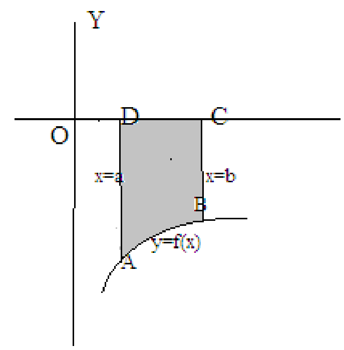
APPLICATIONS OF INTEGRATION

(A) KEY CONCEPTS

1. AREA LYING BELOW THE X-AXIS:



If $f(x)\le 0$ for $a\le x\le b$, then the graph of $y=f(x)$ lies below x-axis

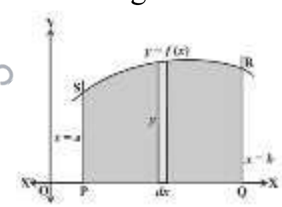
Therefore area bounded by the curve $y=f(x)$, x-axis and the ordinates $x=a$ and $x=b$ is given by

Area ABCD$=\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$

2. AREA LYING ABOVE THE X-AXIS:

The area enclosed by the curve $y= f(x)$, x-axis & between the ordinate at $x=a$ & $x=b$ is given by

$$\text{Area} = \left|\int_a^b y\,dx\right| = \left|\int_a^b f(x)\,dx\right| = \left|F(b) - F(a)\right|$$

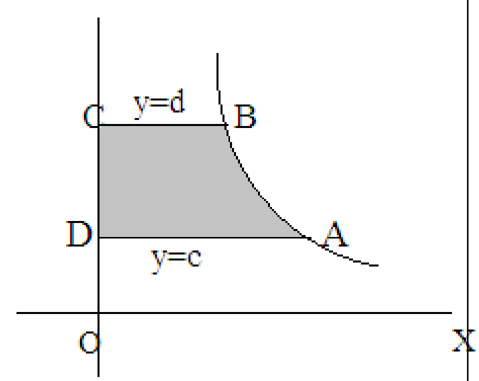


3. AREA LYING ON RIGHT OF Y-AXIS :

Area bounded by the curve $x=f(y)$, y-axis and the abscissa

$y=c$ and $y=d$ is given by

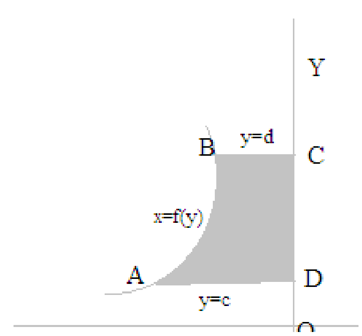
$$\int_c^d x\,dy = \int_c^d f(y)\,dy$$



4. AREA LYING ON LEFT OF Y-AXIS:

The area enclosed by the curve $x= f(y)$, y-axis & between the abscissa at $y=c$ & $y=d$ is given by :

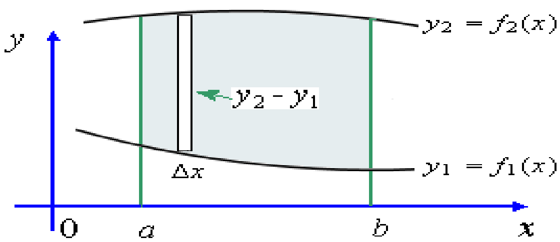
$$\left|\int_c^d x\,dy\right| = \left|\int_c^d f(y)\,dy\right| = \left|F(d) - F(c)\right|$$



5. **AREA BOUNDED BY TWO CURVES**

Area bounded by the two curves y = f(x) & y = g(x) where $f_1(x) \leq f_2(x)$ in $[a, b]$ & between the ordinate x=a & x=b is given by

Area = $\int_a^b (f_2(x) - f_1(x))dx$



IMPORTANT FORMULAE TO USE :

1) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a} + C$

2) Equation of the line in two point form is :

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

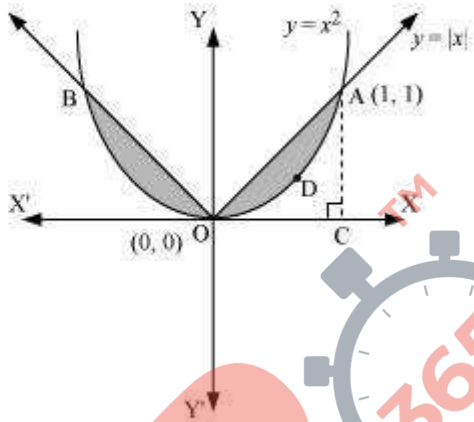
Important Notes

1. If the equation of the curve contains only even powers of x, then the curve is symmetrical about y-axis
2. If the equation of the curve contains only even powers of y, then the curve is symmetrical about x-axis.
3. If the equation of the curve remains unchanged when x is replaced by –x and y by –y, then the curve is symmetrical in opposite quadrants.
4. If the equation of the curve remains unchanged when x and y are interchanged ,then the curve is symmetrical about the line y=x

IMPORTANT BOARD QUESTIONS

1. Find the area of the region $\{(x,y): x^2 \le y \le |x|\}$

Sol. The required area is bounded between two curves $y = x^2$ and $y = |x|$. Both of these curves are symmetric about y-axis and shaded region in the fig. shows the region whose area is required.



Therefore, required area $= 2\times$ area of region R_1

Now to find point of intersection of curves $y = x^2$ and $y = |x|$, we solve them simultaneously.

Clearly, region R_1 is in first quadrant, where $x > 0$

$|x| = x \Rightarrow y = x$……………(i)

$y = x^2$……………(ii)

either $x = 0$ or $x = 1$

The limits are , when x=0, y=0 and when x=1, y=1

So points of intersection of the curve are o(0,0) and A(1,1)

Now, required area $= 2\times$ area of region R_1

$$= 2\int_0^1 [(y \text{ of the line } y = x) - (y \text{ of the parabola } y = x^2)]\,dx$$

$$= 2\int_0^1 (x - x^2)\,dx$$

$$=2\int_0^{1}\left[\frac{\quad}{2}-\frac{\quad}{3}\right]_0^1$$

$$=2\int_0^{1}\left[\frac{1}{2}-\frac{1}{3}\right]$$

$$=\frac{1}{3}\text{ sq. units}$$

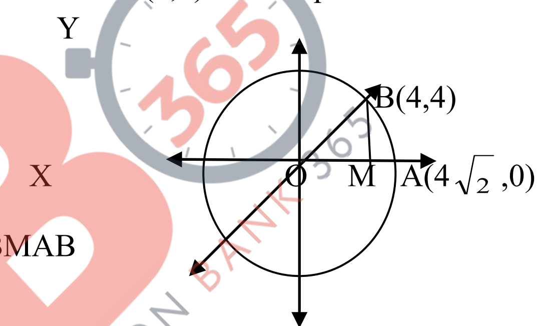
2.Find the area of the region in the first quadrant enclosed by the x-axis , and the line y=x, and the circle $x^2+y^2=32$ (CBSE DELHI 2014)

Sol. The given equations are

y=x…………(i)

and $x^2+y^2=32$ …………..(ii)

Solving (i) and (ii) we find that line and the circle meet at B(4,4) in first quadrant as shown in figure



Draw a perpendicular BM to the x-axis.

Required area =

Area of the region OBMO+ Area of the region BMAB

Now, area of the region OBMO= $\int_0^4 y\,dx$

$$=\int_0^4 x\,dx=\frac{1}{2}[x^2]_0^4 \quad =8\text{…………………(iii)}$$

Again ,area of the region BMAB

$$=\int_4^{4\sqrt{2}} y\,dx$$

$$=\int_4^{4\sqrt{2}}\sqrt{32-x^2}\,dx$$

$$=\left[\frac{1}{2}x\sqrt{32-x^2}+\frac{1}{2}\times 32\times\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$= 8\pi-(8-4\pi)=4\pi-8$……………..(iv)

Adding (iii) and (iv) we get the required area = 4π sq. units

03. Sketch the region bounded by the curve $y\sqrt{5-x^2}$ and $|x-1|$ and find its area using integration

Sol. Consider the given equation y= $\sqrt{5-x^2}$

This equation represents a semicircle with centre at the origin and radius $=\sqrt{5}$ *units*

Given that the region is bounded by the above semicircle and the line y = x − 1

Let us find the point of intersection of the given curve meets the line y= x − 1

$\Rightarrow\sqrt{5-x^2} = |x-1|$

Squaring both the sides, we have,

$$5-x^2 = x^2+1-2x$$

solving the above equation we get $x=-1, \quad x=2$

When x= -1 , y= 2 and when x=2 , y=1

Consider the following figure.

Thus the intersection points are (−1,2) and (2,1)

Consider the following sketch of the bounded region

Required Area, A=

$$\int_{-1}^{2}(y_2-y_1)\,dx$$

$$=\int_{-1}^{1}\sqrt{5-x^2}\,dx+\int_{-1}^{1}(x-1)\,dx+\int_{1}^{2}\left[\sqrt{5-x^2}-(x-1)\right]dx$$

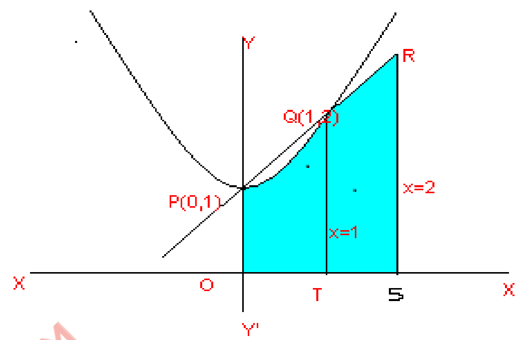
$$=\left[\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\left[\frac{x}{\sqrt{5}}\right]\right]_{-1}^{1}+\left[\frac{x^2}{2}\right]_{-1}^{1}-[x]_{-1}^{1}$$

$$=\frac{5}{2}\sin^{-1}\left[\frac{1}{\sqrt{5}}\right]+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}-\frac{1}{2}$$

$$Required\ area=\left[\frac{5}{2}\sin^{-1}\left[\frac{1}{\sqrt{5}}\right]+\frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}-\frac{1}{2}\right]$$

4. Find the $\{x, y : 0 \le y \le x^2+1, 0 \le y \le x+1, 0 \le x \le 2\}$ **area of region**

Sol.



For getting point of intersection from y=x^2+1 & y=x+1, i.e., P & Q

Required area = shaded region OPQRSTO

= area of the region OTQPO + area(TSRQT)

$= \int_0^1 (x^2+1)\,dx + \int_1^2 (x+1)\,dx$ = 23/6sq. units

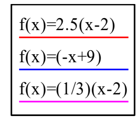
5. Using integration, find area of ΔABC whose vertices have coordinates $A(2,0), B(4,5)$ and $C(6,3)$. As the stats of country are not in equal to each other area wise, do you advise to divide the whole country in the states of equal areas for better development of our nation (VALUE BASED).

Sol. Equation of AB is y = $\frac{5}{2}(x-2)$,

Equation of BC is y = -x + 9,

Equation of AC is y = $\frac{3}{4}(x-2)$

Required area = $\int_2^4 \frac{5}{2}(x-2)\,dx + \int_4^6 (-x+9)\,dx - \int \frac{3}{9}(x-2)\,dx$ =7 square unit



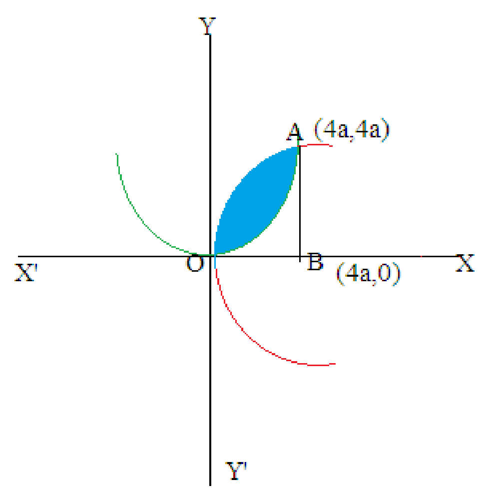
VALUE: We should not divide our country in states of equal areas because states are divided as per their culture and heritage which is an essential part for a well being

6. Find the area of region included between the parabolas y^2=4ax and x^2=4ay,where a>0

Sol. Given parabolas y^2=4ax ……………….. (i)

and x^2=4ay……………………………………(ii)

Finding the points of intersection at point A we solve



$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

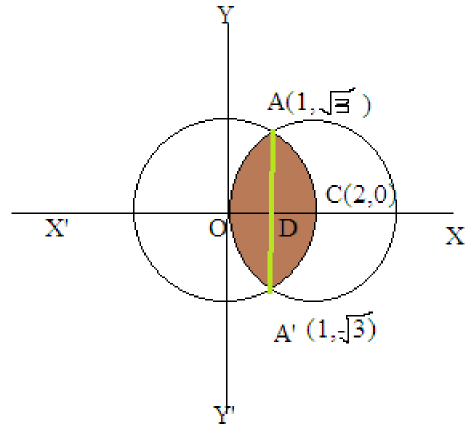
We get x=4a and y=4a

Required area=Area of shaded region

$$=\int_0^{4a}\sqrt{4ax}\,dx - \int_0^{4x}\frac{x^2}{4a}dx$$

On solving we get required area $=\frac{16a^2}{3}$ sq. units

7. Find the area of region enclosed between two circles : $x^2+y^2=4$ and $(x-2)^2+y^2=4$



Sol. Given equation of circles are

$x^2+y^2=4$ …………………(i)

$(x-2)^2+y^2=4$………………(ii)

Solving equation (i) and (ii)

$x^2+y^2= (x-2)^2+y^2$

$x^2-4x+4+ y^2= x^2+y^2$

$x=1$ and $y=\pm\sqrt{3}$

Required area of the enclosed region OACA'O between circles=2(area of the region ODCAO)

=2(area of the region ODAO+ area of the region DCAD)

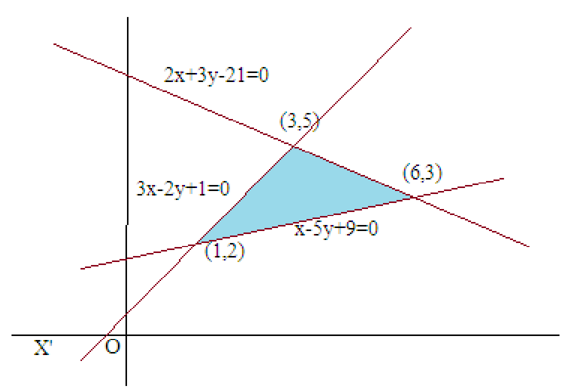
$$=2\left[\int_0 ydx + \int_1^2 ydx\right]$$

$$=2\left[\int_0^2 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right]$$

On solving and applying formula of integration we get $\left[\left(-\sqrt{3}-4\times\frac{\pi}{6}\right)+4\times\frac{\pi}{2}\right]+\left[\frac{4\pi}{2}-\sqrt{3}-4\times\frac{\pi}{6}\right]$

Getting answer as $\frac{8\pi}{3}-2\sqrt{3}\,squni\,ts$

8. Using the method of integration , find the area of the region bounded by the lines 3x-2y+1=0,2x+3y-21=0 and x-5y+9=0



Sol. $3x-2y+1=0$ ………………… (i)

$2x+3y-21=0$ ………………… (ii)

$x-5y+9=0$ …………………(iii)

On solving (i) and (ii) we get

$x=3$ and $y=5$

On solving (ii) and (iii) we get

x=6 and ,y=3

similarly point of intersection of the lines (i) and (iii) is (1,2)

Area of required region $=\int_1^3 \frac{3x+1}{2}dx + \int_3^6 \frac{-2x+21}{3}dx - \int_1^6 \frac{x+9}{5}dx$

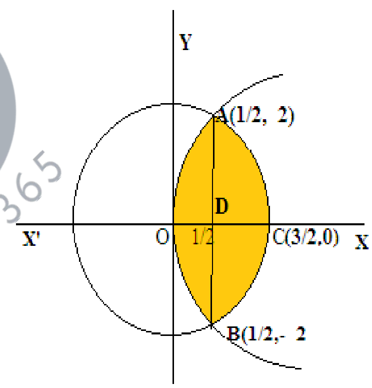
$$=\frac{3}{2}\left[\frac{x^2}{2}\right]_1^3 + \frac{1}{2}[x]_1^3 - \frac{2}{3}\left[\frac{x^2}{2}\right]_3^6 + 7[x]_3^6 - \frac{1}{5}\left[\frac{x^2}{2}\right]_1^6 - \frac{9}{5}[x]_1^6$$

=13/2 sq. units

9. Using the method of integration, find the area of the region bounded by the parabola $y^2=4x$ and the circle $4x^2+4y^2=9$.

Sol. Given curves are $y^2=4x$ ……………………(i)

$4x^2+4y^2=9$ …………………(ii)



Finding the points of intersection points A and B

From equation (i) and (ii)

$4x^2+16x-9=0$

Solving by splitting the middle term we get x=1/2 or -9/2(not possible)

We get x=1/2 and y= +2 and -2.

Figure

Area of the required region =2[Area OADO +Area DACD]

$$=2\left[\int_0^{1/2} 2\sqrt{x}\,dx + \int_{1/2}^{3/2}\sqrt{\frac{9}{4}-x^2}\,dx\right]$$

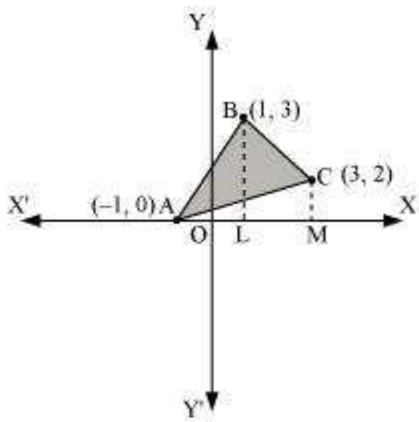
$$=4\cdot\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^{1/2} + \left[\frac{x}{2}\sqrt{\frac{9}{4}-x^2} + \frac{1}{2}\cdot\frac{9}{4}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{1/2}^{3/2}$$

figure

Solving and getting $\frac{2\sqrt{2}}{3} - \frac{1}{\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$ as answer.

10. Using integration find the area of region bounded by the triangle whose verticesare (– 1, 0), (1, 3) and (3, 2).

Sol.



BL and CM are drawn perpendicular to x-axis.

It can be observed in the following figure that,

Area (ΔACB) = Area (ALBA) + Area (BLMCB) – Area (AMCA) … (1)

Equation of line segment AB is

$$y-0=\frac{3-0}{1+1}(x+1)$$
$$y=\frac{3}{2}(x+1)$$
$$\therefore \text{Area}(\text{ALBA})=\int_{-1}^{1}\frac{3}{2}(x+1)dx=\frac{3}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}=\frac{3}{2}\left[\frac{1}{2}+1-\frac{1}{2}+1\right]=3\text{ units}$$

Equation of line segment BC is

$$y-3=\frac{2-3}{3-1}(x-1)$$
$$y=\frac{1}{2}(-x+7)$$
$$\therefore \text{Area}(\text{BLMCB})=\int_{1}^{3}\frac{1}{2}(-x+7)dx=\frac{1}{2}\left[-\frac{x^2}{2}+7x\right]_{1}^{3}=\frac{1}{2}\left[-\frac{9}{2}+21+\frac{1}{2}-7\right]=5\text{ units}$$

Equation of line segment AC is

$$y-0=\frac{2-0}{3+1}(x+1)$$

$$y=\frac{1}{2}(x+1)$$

$$\therefore \text{Area}(\text{AMCA})=\frac{1}{2}\int_{-1}^{3}(x+1)\,dx=\frac{1}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{3}=\frac{1}{2}\left[\frac{9}{2}+3-\frac{1}{2}+1\right]=4 \text{ units}$$

Therefore, from equation (1), we obtain

Area (ΔABC) = (3 + 5 – 4) = 4 units

HOTS

1.Using integration,find the area of the region bounded by the curves $y = \sqrt{4 - x^2}$ and $x^2 + y^2 - 4x = 0$ and the x-axis.

Ans. $\left(\frac{4\pi}{3} - \sqrt{3}\right)$ sq units

[Hint: Point of intersection of circles is $(1, \sqrt{3})$]

2. If the area bounded by parabola $y^2 = 16ax$ and the line y= 4mx is $\frac{a^2}{12}$,then using integration, find m.

Ans. m=2√2

[Hint: Required area = $\int_0^{\frac{a}{m^2}} \sqrt{16ax}\,dx - \int_0^{\frac{a}{m^2}} 4mx\,dx$]

3. Using integration,find the area bounded by the curve $y = |x - 1|$ and $y = 3 - |x|$

Ans. $4\frac{1}{2}$ sq. units

Hint: Required area = $\int_{-1}^{2} (3 - |x|)\,dx - \int_{-1}^{2} (x - 1)\,dx$]

4. Find the area of that part of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2=6x$.

Ans. $\frac{4}{3}(\sqrt{3} + 10\pi)$sq. units

Hint: $\int_0^{2\sqrt{3}} \left[\sqrt{16 - y^2} - \frac{y^2}{6}\right] dy$

5.Prove that the curves $y^2=4x$ and $x^2=4y$ divide the area of the square bounded by x=0, x=4, y=0 and y=0 into three equal parts.

6. Find the area bounded by curve x=3cost, y=2sint

Ans: 6π sq. units.

[Hint: Parametric form of the given function is $\left[\frac{x^2}{9}+\frac{y^2}{4}=1\right]$.

7. Using integration ,find the area of the following region: $\{(x,y): |x+2| \le y \le \sqrt{20-x^2}$

Ans. ($5\pi - 2$) sq units

Hint: $\int_{-4}^{2}\sqrt{20-x^2}\,dx - -2\int_{-4}^{-2} -(x-2)dx - \int_{-2}^{2}(x+2)dx$