Differential equation

Key Concepts

Definition: An equation involving the independent variable x(say), dependent variable y(say) and the differential coefficient of dependent variable w.r.t independent variable i.e $\frac{dy}{dx}, \frac{d^2y}{dx^2}$,.... etc is called differential equation.

Order of a differential equation is the order of the highest order derivative occurring in the differential equation.

Degree of a differential equation is the degree of highest order derivative occurring in the differential equation when the differential coefficients are made free from radicals, fractions and it is written as a polynomial in differential coefficients.

e.g $\left(\frac{d^2y}{dx^2}\right)^3 + sin\left(\frac{dy}{dx}\right) = 0$ here order is 2 but this differential equation can't be written in the form of polynomial in differential coefficient Hence its degree not defined.

Linear and Nonlinear Differential Equations: A differential equation in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together, is called a linear differential equation otherwise it is non-linear.

"Formation of differential Equation" To form a DE from a given equation in x and y containing arbitrary constants (parameters) –

"**Initial value problem(IVP)** is one in which some initial conditions are given to solve a DE"

1. Differentiate the given equation as many times as the number of arbitrary constants involved in it.

2. Eliminate the arbitrary constant from the equations of y, y', y'' etc.

Solution of Differential Equations-

1. **Variable separable form**
2. **Homogenous Equations**
3. **Linear Differential Equations**

VARIABLE SEPARABLE FORM If in the equation, it is possible to get all terms containing x and dx to one side and all the terms containing y and dy to the other, the variables are said to be separable,

Procedure to solve:

Consider the equation $\frac{dy}{dx}$ = X.Y, where X is a function of x only and Y is function of y only.

(i) Put the equation in the form $\frac{1}{Y}$,dy=X.dx.

(ii) Integrating both the sides we get

$\int \frac{dy}{Y} = \int X dx + C$ where C is an arbitrary constant.

Thus the required solution is obtained.

Homogeneous Differential Equations

A differential equation which can be expressed in the form $\frac{dy}{dx}$ = f(x,y) or $\frac{dx}{dy}$=g(x,y) where, f (x, y) and g(x, y) are homogenous functions of degree zero is called a homogeneous differential equation

Steps for Solving a Homogeneous Differential Equation

1. Rewrite the differential in homogeneous form

2. Make the substitution y = vx orx = vy where v is a variable.

3. Substitute to rewrite the differential equation in terms of v and x or v and y only

4. Follow the steps for solving separable differential equations.

5. Re-substitute v = y/x or v = x / y in the final solution.

Linear Differential Equation: A first-order linear differential equation can be written in the form $\frac{dy}{dx} + Py = Q$)where P and Q are constants or function of x only**or** $\frac{dx}{dy}$ **+Px = Q**where P and Q are constants or function of y only.

IMPORTANT BOARD QUESTIONS

1. Solve the following differential equation:

$$\sqrt{1 + x^2 + y^2 + x^2y^2} + xy\frac{dy}{dx} = 0$$

Solution. $\sqrt{1 + x^2 + y^2(1 + x^2)} = -xy\frac{dy}{dx}$

$\int \frac{ydy}{\sqrt{1+y^2}} = -\int \frac{\sqrt{1+x^2}}{x^2}.xdx$ 1mark

Let $1 + y^2 = u^2$ & $1 + x^2 = v^2$

$$\int \frac{u\,du}{u} = -\int \frac{v^2}{v^2-1}dv = -\int \frac{(v^2-1)+1}{v^2-1}dv = -\int \left(1 + \frac{1}{v^2-1}\right)dv$$

$u = -v - \frac{1}{2}log\left|\frac{v-1}{v+1}\right| + c.$ 2marks

$\sqrt{1+y^2} = -\sqrt{1+x^2} - \frac{1}{2}log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + c.$ 1mark

2. Find the general solution of the differential equation

$$(x^2 + 1)\frac{dy}{dx} + 2xy = \sqrt{x^2 + 4}$$

Solution: $\frac{dy}{dx} + \left(\frac{2x}{x^2+1}\right)y = \frac{\sqrt{x^2+4}}{x^2+1}$, (compare it with $\frac{dy}{dx} + Py = Q$)

1mark

$IF = e^{\int pdx} = e^{\int \frac{2x}{x^2+1}dx} = e^{log|x^2+1|}$=$x^2+1$ 1mark

Reqd sol is y(x^2+1)=$\int \frac{\sqrt{x^2+4}}{x^2+1}$.($x^2$+1)dx

⇒y(x^2+1)=$\int \sqrt{x^2+4}$dx

y(x^2+1) =$\frac{x}{2}\sqrt{x^2+4} + 2log\left|x+\sqrt{x^2+4}\right| + c.$ 2marks

3. Find the particular solution of differential equation: $(x^2 - y^2)\, dx + 2\, xydy = 0$. Given that $y = 1$ when $x = 1$

Solution:

$$\left(x^2-y^2\right)dx+2xydy=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^2-x^2}{2xy} \qquad ...(1)$$

It is a homogeneous differential equation.

$$\text{Let } y=vx \qquad ...(2)$$

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx} \qquad ...(3)$$

Substituting (2) and (3) in (1), we get:

$$v+x\frac{dv}{dx}=\frac{v^2x^2-x^2}{2x(vx)}$$

$$v+x\frac{dv}{dx}=\frac{x^2\left(v^2-1\right)}{2vx^2}=\frac{v^2-1}{2v}$$

$$2v^2+2vx\frac{dv}{dx}=v^2-1$$

$$2vx\frac{dv}{dx}=-v^2-1$$

$$\left(\frac{2v}{v^2+1}\right)dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{2v}{v^2+1}dv = -\int\left(\frac{1}{x}\right)dx$$

$$\log\left|v^2+1\right| = -\log|x| + \log C$$

$$\log\left|v^2+1\right| = \log\left|\frac{C}{x}\right|$$

$$v^2+1 = \frac{C}{x}$$

$$x\left(v^2+1\right) = C$$

$$x\left[\left(\frac{y}{x}\right)^2+1\right] = C$$

$$y^2+x^2 = Cx \qquad ...(4)$$

It is given that when $x = 1, y = 1$
$(1)^2 + (1)^2 = C\ (1)$
$\Rightarrow C = 2$

Thus, the required equation is $y^2 + x^2 = 2x$.

4. Solve the differential equation : $x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$

Solution: This is a linear differential equation

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2} \qquad \text{...(1)}$$

$$I.F = e^{\int\frac{1}{x\log x}dx} = \log x \qquad \text{...............................(1)}$$

y(logx)=$\int\frac{2}{x^2}logxdx$+C

y(logx) = $2logx\int\frac{1}{x^2}dx - \int\left[\frac{d}{dx}(logx).\int\frac{1}{x^2}dx\right]dx$ ....(1)

y(logx) = $-2\frac{logx}{x} - \frac{1}{x} + C$ ....,(1)

5. Form the differential equation of the family of circles having centre on the y −axis and radius 3 units.

Solution: Equation of circle having centre(0,a) and radius is 3 units

$x^2 + y^2 - 2ay + a^2 - 9 = 0$ .... (1) {using $(x\text{-}h)^2+(y\text{-}k)^2=r^2$}

- Differentiating: $2x + 2yy' - 2ay' = 0$
- Getting $a = \frac{x+yy'}{y'}$
- Putting the value of a in (1) to get the answer

6. Find the general solution of the differential equation $2x^2\frac{dy}{dx}-2xy+y^2=0$.

Solution: Consider the equation $2x^2\frac{dy}{dx} - 2xy + y^2 = 0$

$\frac{dy}{dx}=\frac{2xy-y^2}{2x^2}$ ....(1)

Let y=vx ; $\frac{dy}{dx}$ = v + x $\frac{dv}{dx}$

From (i), we get

v + x $\frac{dv}{dx}$= v- $\frac{v^2}{2}$

x $\frac{dv}{dx}$= - $\frac{v^2}{2}$

Integrating both sides by separating the variable

$-\frac{1}{v} = -\frac{1}{2}$ log | x | + c

$-\frac{x}{y} = -\frac{1}{2}$ log | x | + c is the required solution.

7. Form the differential equation representing the parabola having vertex at the origin and axis along positive direction of x-axis.

Solution: Given y^2= 4ax ………………………………1

$2y\frac{dy}{dx}=4a$ ………………………………1

$y\frac{dy}{dx}=2\frac{y^2}{4x}$ ………………………………1

$y\frac{dy}{dx}=\frac{y^2}{2x}$ which is the required differential equation. ……1

8. Solvethe$(x+3y^2)\frac{dy}{dx}$=y (y>0)

Solution: Given differential equation is $(x+3y^2)\frac{dy}{dx}$=y (y>0)

We can write as $\frac{dx}{dy}=\frac{x+3y^2}{y}=\frac{1}{y}.x+3y \Rightarrow \frac{dx}{dy}+(\frac{-1}{y})$ .x=3y

This is a linear equation in the form

$\frac{dx}{dy}$+Px=Q where P = $-\frac{1}{y}$ and Q is $3y$

IF $=e^{\int pdy}=e^{\int -\frac{1}{y}dy}=e^{-\log y}=-\frac{1}{y}$

Required solution is $x\times IF=\int Q\times IFdy+C$

9. Solve $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$

Solution: $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$

$\Rightarrow e^{\int pdy}=e^{\int -\frac{1}{y}dy}\Rightarrow e^y(x^2+e^x)dx$

Integrating both sides, we get

$\Rightarrow \int e^y dy=\int (e^x+x^2)dx$

$\Rightarrow e^{y} = e^{x} + \frac{x^3}{3} + C$, which is the required solution.

10. Solve $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

Solution: we have $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

$\Rightarrow \frac{dx}{dy} = \frac{-e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}} = g\left(\frac{x}{y}\right)$ ……………………………………1

Here, RHS of differential equation is of the form $g\left(\frac{x}{y}\right)$, so it is a homogenous function of degree zero.

Now we put x= vy and, $\left(\frac{dx}{dy}\right) = v + y\frac{dv}{dy}$

From 1, we get $v + y\frac{dv}{dy} = \frac{-e^{v}(1 - v)}{1 + e^{v}}$

$y\frac{dv}{dy} = \frac{-e^{v}(1 - v)}{1 + e^{v}} - v = -\left(\frac{(v + e^{v})}{1 + e^{v}}\right) \Rightarrow \frac{1 + e^{v}}{-(v + e^{v})}dv = \frac{dy}{y}$

On integrating both sides, we get

$-\log\left|v + e^{v}\right| + \log C = \log\left|y\right| \Rightarrow \log C = \log\left|y\right| + \log\left|v + e^{v}\right|$

$\Rightarrow$ C= $y(v + e^{v})$

On substituting value of v, we get $x + ye^{\frac{x}{y}} = C$, which is required solution.

HOTS

1. Solve $(x^2 - y^2)dx + 2xydy = 0$, given that $y = 1$, when $x = 1$.

2. Write the order and degree of differential equation $\left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}} = \left(y + \frac{dy}{dx}\right)^{\frac{1}{2}}$

3. Find the particular solution of differential equation

$\frac{dy}{dx} + y\cot x = 2x + x^2\cot x$, given that $y\left(\frac{\pi}{2}\right) = 0$.

4.Find the particular solution of differential equation$\sqrt{1 - y^2}$dx= $(\sin^{-1}y - x)$dy

5.Form the differential equation representing the family of ellipses having foci on x –axis and centre at origin.

6.Write the order and degree of the differential equation $\left(\frac{d^3y}{dy^3}\right)^2 + \frac{d^2y}{dx^2} + \sin\left(\frac{dy}{dx}\right) = 0$

Hints

1.$\frac{dy}{dx} = \frac{x^2 - y^2}{-2xy}$which is homogenous differential equation. Put y = vx $\Rightarrow \frac{dy}{dx}$= v+x$\frac{dv}{dx}$

solution.

*using given initial values, find the value of integrating constant C.

2. Order 2 and degree 4

3. Given equation is a linear differential equation. Compare it with $\frac{dy}{dx}+Py=Q$. We get P=cotx and Q= $2x+x^2$cotx,

IF = $e^{\int pdx}$

General solution is y.IF= $\int QIFdx + C$

4. $\frac{dx}{dy}=\frac{\sin^{-1}y}{\sqrt{1-y^2}}-\frac{x}{\sqrt{1-y^2}}$

$\frac{dx}{dy}+\frac{x}{\sqrt{1-y^2}}=\frac{\sin^{-1}y}{\sqrt{1-y^2}}$

Which is first order linear differential equation of the form $\frac{dx}{dy}+Px=Q$

Find IF= $e^{\int pdy}$

General Solution be x.IF= $\int(Q.IF)dy + C$

5.Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Now diff, it two times w.r.t x and eliminate a & b

6.Order is 3 and degree not defined .