CBSE Board
Class XII Mathematics
Board Paper 2008
Delhi Set – 1

Time: 3 hrs **Total Marks: 100**

General Instructions:

1. All questions are compulsory.
2. The question paper consists of 29 questions divided into three Section A, B and C. Section A comprises of 10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section C comprises of 7 questions of six marks each.
3. All questions in section A are to be answered in one word, one sentence or as per the exact requirement of the question.
4. There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
5. Use of calculators is not permitted. You may ask for logarithmic tables, if required.

SECTION – A

1. If f(x) = x + 7 and g(x) = x – 7, x $\in$ R, find (fog) (7)

2. Evaluate: $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(\frac{1}{2}\right)\right]$

3. Find the value of *x* and *y* if: $2\begin{bmatrix}1 & 3\\ 0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

4. Evaluate: $\begin{vmatrix}a+ib & c+id\\ -c+id & a-ib\end{vmatrix}$

5. Find the co-factor of a_{12} in the following:

$$\begin{vmatrix}2 & -3 & 5\\ 6 & 0 & 4\\ 1 & 5 & -7\end{vmatrix}$$

6. Evaluate: $\int \frac{x^2}{1+x^3}dx$

7. Evaluate: $\int_0^4 \frac{dx}{1+x^2}dx$

8. Find a unit vector in the direction of $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$

9. Find the angle between the vectors $\vec{a}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}-\hat{k}$

10. For what value of λ are the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ perpendicular to each other?

SECTION – B

11. (i) Is the binary operation *, defined on set N, given by $a*b=\frac{a+b}{2}$ for all $a,b\in N$, commutative?

(ii) Is the above binary operation * associative?

12. Prove the following:

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$$

13. Let $A=\begin{bmatrix}3&2&5\\4&1&3\\0&6&7\end{bmatrix}$. Express A as sum of two matrices such that one is symmetric and the other is skew symmetric.

OR

If $A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$, verify that $A^2-4A-5I=0$.

14. For what value of k is the following function continuous at x = 2?

$$f(x)=\begin{cases}2x+1 & ; \quad x<2\\ k & ; \quad x=2\\ 3x-1 & ; \quad x>2\end{cases}$$

15. Differentiate the following with respect of x:

$$y=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

16. Find the equation of tangent to the curve x = sin 3t, y = cos 2t, at $t = \frac{\pi}{4}$

17. Evaluate: $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

18. Solve the following differential equation:
$(x^2 - y^2)\, dx + 2xy\, dy = 0$
given that y = 1 when x = 1

OR

Solve the following differential equation:
$\frac{dy}{dx} = \frac{x\ 2y - x}{x\ 2y + x}$, if y = 1 when x = 1

19. Solve the following differential equation:
$\cos^2 x \frac{dy}{dx} + y = \tan x$

20. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a}.\vec{c} = 3$

OR

If $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7$, show that the angle between $\vec{a}$ and $\vec{b}$ is 60^o.

21. Find the shortest distance between the following lines:
$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$

OR

Find the point on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance $3\sqrt{2}$ from the point (1, 2, 3).

22. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of the number of successes.

SECTION – C

23. Using properties of determinants, prove the following:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = \alpha-\beta \quad \beta-\gamma \quad \gamma-\alpha \quad \alpha+\beta+\gamma$$

24. Show that the rectangle of maximum area that can be inscribed in a circle is a square.

OR

Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height *h* is $\frac{1}{3}h$.

25. Using integration find the area of the region bounded by the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$.

26. Evaluate: $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx$

27. Find the equation of the plane passing through the point (–1, – 1, 2) and perpendicular to each of the following planes:
$2x + 3y - 3z = 2$ and $5x - 4y + z = 6$

OR

Find the equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$

28. A factory owner purchases two types of machines, A and B for his factory. The requirements and the limitations for the machines are as follows:

Machine	Area occupied	Labour force	Daily output (in units)
A	1000 m^2	12 men	60
B	1200 m^2	8 men	40

He has maximum area of 9000 m^2 available, and 72 skilled labourers who can operate both the machines. How many machines of each type should he buy to maximise the daily output?

29. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter, a car and a truck are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

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SECTION – A

1. fog(x) = f{g(x)}
= f (x – 7)
= {(x – 7) + 7}
= x
∴fog (7) = 7

2. We know that the domain and range of the principal value branch of function $\sin^{-1}$ is defined as:

$$\sin^{-1}:[-1,1]\rightarrow\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\therefore \sin\left[\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right]=\sin\left[\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]$$

$$=\sin\left[\frac{\pi}{2}\right]$$

$$=1$$

3.

$$2\begin{bmatrix}1 & 3\\ 0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$$

$$\begin{bmatrix}2 & 6\\ 0 & 2x\end{bmatrix}+\begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$$

$$\begin{bmatrix}2+y & 6+0\\ 0+1 & 2x+2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$$

$$\begin{bmatrix}2+y & 6\\ 1 & 2(x+1)\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$$

On comparing the corresponding elements of the matrices on both sides, we get:
2 + y = 5 ⇒ y = 3
2(x + 1) = 8 ⇒ x = 3

4.

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} = (a + ib)(a - ib) - (c + id)(-c + id)$$

$= (a^2 - i^2b^2) - (-c^2 + i^2d^2)$

$= (a^2 + b^2) - (-c^2 - d^2) \quad (\because i^2=-1)$

$= a^2 + b^2 + c^2 + d^2$

5.

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

Cofactor of $a_{12} = (-1)^{1+2}[6(-7) - 4(1)] = (-1)[-42 - 4] = 46$

6. $\int \frac{x^2}{1+x^3}dx$

Let $1 + x^3 = t$

$\Rightarrow 0 + 3x^2dx = dt$

$\Rightarrow x^2dx = \frac{dt}{3}$

$\therefore \int \left(\frac{x^2}{1+x^3}\right)dx = \int \frac{\frac{dt}{3}}{t}$

$= \frac{1}{3}\int \frac{dt}{t}$

$= \frac{1}{3}\log|t| + c$

$= \frac{1}{3}\log|1+x^3| + c$

7. $\int_0^4 \frac{dx}{1+x^2} dx$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$dx = \sec^2\theta \; d\theta$

When $x = 0$, $\theta = \tan^{-1}(0) = 0$

When $x = 1$, $\theta = \tan^{-1}1 = \frac{\pi}{4}$

$$\therefore \int_0^4 \frac{dx}{1+x^2} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^2\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta$$

$$= [\theta]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{4}$$

8. The unit vector $(\hat{a})$ in the direction of $\vec{a}$ is given by $\frac{\vec{a}}{|\vec{a}|}$

$$\hat{a} = \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + -2^2 + 6^2}}$$

$$= \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{9+4+36}}$$

$$= \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{49}}$$

$$= \frac{1}{7}\left[3\hat{i} - 2\hat{j} + 6\hat{k}\right]$$

$$= \frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$$

9. $\vec{a}=\hat{i}-\hat{j}+\hat{k}$

$\vec{b}=\hat{i}+\hat{j}-\hat{k}$

Now, $\vec{a}.\vec{b}=\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$

Where θ is the angle between $\vec{a}$ and $\vec{b}$; also $0\le\theta\le\pi$

$$\hat{i}-\hat{j}+\hat{k}\ .\ \hat{i}+\hat{j}-\hat{k}\ =\left(\sqrt{1\ ^2+\ -1\ ^2+\ 1\ ^2}\right)\left(\sqrt{1\ ^2+\ 1\ ^2+\ -1\ ^2}\right)\cos\theta$$

$$\left[\ 1\ \ 1\ +\ -1\ \ 1\ +\ 1\ \ -1\ \right]=\left[\sqrt{3}\sqrt{3}\cos\theta\right]$$

$$1-1-1\ =3\cos\theta$$

$$-1=3\cos\theta$$

$$\cos\theta=\frac{-1}{3}\Rightarrow\theta=\cos^{-1}\left(\frac{-1}{3}\right)$$

10. $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$

$\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$

If $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then $\vec{a}.\vec{b}$ must be 0.

$$\vec{a}.\vec{b}=\ 2\hat{i}+\lambda\hat{j}+\hat{k}\ .\ \hat{i}-2\hat{j}+3\hat{k}$$

$$0=\ 2\ \ 1\ +\lambda\ \ -2\ +\ 1\ \ 3$$

$$0=2-2\lambda+3$$

$$2\lambda=5\Rightarrow\lambda=\frac{5}{2}$$

Thus, the value of λ is $\frac{5}{2}$.

SECTION – B

11.

(i) For all $a, b \in N, a * b = \frac{a+b}{2}$

Now, $b * a = \frac{b+a}{2} = \frac{a+b}{2} = a * b$

Thus, the binary operation * is commutative.

(ii) Let a,b,c $\in$ N

$$a * \ b * c \ = a * \left(\frac{b+c}{2}\right) = \frac{a + \frac{b+c}{2}}{2} = \frac{2a+b+c}{4}$$

$$a * b \ * c = \left(\frac{a+b}{2}\right) * c = \frac{\frac{a+b}{2} + c}{2} = \frac{a+b+2c}{4}$$

$\therefore \ a * b \ * c \neq a * \ b * c$

Thus, the binary operation * is not associative.

12. L.H.S. $= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$$= \tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{5}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{5}\right)}\right] + \tan^{-1}\left[\frac{\frac{1}{7}+\frac{1}{8}}{1-\left(\frac{1}{7}\right)\left(\frac{1}{8}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{5+3}{15}}{\frac{15-1}{15}}\right] + \tan^{-1}\left[\frac{\frac{8+7}{56}}{\frac{56-1}{56}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{8}{15}}{\frac{14}{15}}\right] + \tan^{-1}\left[\frac{\frac{15}{56}}{\frac{55}{56}}\right]$$

$$= \tan^{-1}\left(\frac{8}{14}\right) + \tan^{-1}\left(\frac{15}{55}\right)$$

$$= \tan^{-1}\left(\frac{4}{7}\right) + \tan^{-1}\left(\frac{3}{11}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{7}+\frac{3}{11}}{1-\left(\frac{4}{7}\right)\left(\frac{3}{11}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{44+21}{77}}{\frac{77-12}{77}}\right)$$

$$= \tan^{-1}\left(\frac{65}{65}\right)$$

$$= \tan^{-1} 1$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4}$$

$=$ R.H.S.

Hence proved.

13. $A=\begin{bmatrix}3 & 2 & 5\\ 4 & 1 & 3\\ 0 & 6 & 7\end{bmatrix}$

$A'=\begin{bmatrix}3 & 4 & 0\\ 2 & 1 & 6\\ 5 & 3 & 7\end{bmatrix}$

Now, A can be written as:

$A = \frac{1}{2}\ A+A'\ +\frac{1}{2}\ A-A'$

$$A+A'=\begin{bmatrix}3 & 2 & 5\\ 4 & 1 & 3\\ 0 & 6 & 7\end{bmatrix}+\begin{bmatrix}3 & 4 & 0\\ 2 & 1 & 6\\ 5 & 3 & 7\end{bmatrix}$$

$$=\begin{bmatrix}3+3 & 2+4 & 5+0\\ 4+2 & 1+1 & 3+6\\ 0+5 & 6+3 & 7+7\end{bmatrix}$$

$$=\begin{bmatrix}6 & 6 & 5\\ 6 & 2 & 9\\ 5 & 9 & 14\end{bmatrix}$$

$$\frac{1}{2}\ A+A'\ =\frac{1}{2}\begin{bmatrix}6 & 6 & 5\\ 6 & 2 & 9\\ 5 & 9 & 14\end{bmatrix}=\begin{bmatrix}3 & 3 & \frac{5}{2}\\ 3 & 1 & \frac{9}{2}\\ \frac{5}{2} & \frac{9}{2} & 7\end{bmatrix}=P\text{, say}$$

$$\text{Now, } P'=\begin{bmatrix}3 & 3 & \frac{5}{2}\\ 3 & 1 & \frac{9}{2}\\ \frac{5}{2} & \frac{9}{2} & 7\end{bmatrix}$$

Thus, $P = \frac{1}{2}\ A+A'$ is a symmetric matrix.

Now, $A-A' = \begin{bmatrix} 3-3 & 2-4 & 5-0 \\ 4-2 & 1-1 & 3-6 \\ 0-5 & 6-3 & 7-7 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix}$

$\frac{1}{2}\left(A-A'\right) = \begin{bmatrix} 0 & -1 & \frac{5}{2} \\ 1 & 0 & \frac{-3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 0 \end{bmatrix} = Q$, say

Now, $Q' = \begin{bmatrix} 0 & 1 & -\frac{5}{2} \\ -1 & 0 & \frac{3}{2} \\ \frac{5}{2} & -\frac{3}{2} & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & \frac{5}{2} \\ 1 & 0 & \frac{-3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 0 \end{bmatrix} = -Q$

Thus, $Q = \frac{1}{2}\left(A-A'\right)$ is a skew symmetric matrix.

$\therefore A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 3 & \frac{5}{2} \\ 3 & 1 & \frac{9}{2} \\ \frac{5}{2} & \frac{9}{2} & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1 & \frac{5}{2} \\ 1 & 0 & \frac{-3}{2} \\ \frac{-5}{2} & \frac{3}{2} & 0 \end{bmatrix}$

OR

$$A^2=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$$

$$=\begin{bmatrix}1\times1+2\times2+2\times2&2\times1+2\times1+2\times2&1\times2+2\times2+2\times1\\2\times1+1\times2+2\times2&2\times2+1\times1+2\times2&2\times2+1\times2+2\times1\\2\times1+2\times2+1\times2&2\times2+2\times1+1\times2&2\times2+2\times2+1\times1\end{bmatrix}$$

$$=\begin{bmatrix}1+4+4&2+2+4&2+4+2\\2+2+4&4+1+4&4+2+2\\2+4+2&4+2+2&4+4+1\end{bmatrix}=\begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}$$

$$4A=4\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}=\begin{bmatrix}1\times4&2\times4&2\times4\\2\times4&1\times4&2\times4\\2\times4&2\times4&1\times4\end{bmatrix}=\begin{bmatrix}4&8&8\\8&4&8\\8&8&4\end{bmatrix}$$

$$5I=5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$$

$$A^2-4A-5I=\begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}-\begin{bmatrix}4&8&8\\8&4&8\\8&8&4\end{bmatrix}-\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$$

$$=\begin{bmatrix}9-4-5&8-8-0&8-8-0\\8-8-0&9-4-5&8-8-0\\8-8-0&8-8-0&9-4-5\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O=\text{R.H.S.}$$

14. The given function f(x) will be continuous at x = 2, if

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+1) = 2\times 2+1=5$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x-1) = (3\times 2-1) = 5$$

$$\therefore f(2) = k$$

$$\Rightarrow k=5$$

Thus, for k = 5, the given function is continuous at x = 2.

15. Let x = cos2θ $\Rightarrow \theta = \frac{1}{2}\cos^{-1}x$ ... (1)

$$\therefore \sqrt{1+x} = \sqrt{1+\cos 2\theta} = \sqrt{1+2\cos^2\theta-1} = \sqrt{2}\cos\theta$$

$$\sqrt{1-x} = \sqrt{1-\cos 2\theta} = \sqrt{1-(1-2\sin^2\theta)} = \sqrt{2}\sin\theta$$

$$\text{Let } y = \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\}$$

$$= \frac{\pi}{4}-\theta = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x \qquad [\text{From } (1)]$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{1}{2\sqrt{1-x^2}}$$

16. $x = \sin 3t \Rightarrow \frac{dx}{dt} = 3\cos 3t$

$$\therefore\ x_{\left(t=\frac{\pi}{4}\right)} = \sin 3\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$y = \cos 2t$

$$\Rightarrow \frac{dy}{dt} = -2\sin 2t$$

$$\therefore y_{\left(t=\frac{\pi}{4}\right)} = \cos 2t = \cos 2\left(\frac{\pi}{4}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$= -2\sin 2t\ \frac{1}{3\cos 3t}$$

$$= -\frac{2}{3}\left(\frac{\sin 2t}{\cos 3t}\right)$$

$$\therefore \frac{dy}{dx}_{\left(t=\frac{\pi}{4}\right)} = \frac{-2}{3}\frac{\sin\left(2\times\frac{\pi}{4}\right)}{\cos\left(3\times\frac{\pi}{4}\right)}$$

$$= -\frac{2}{3}\frac{\sin\frac{\pi}{2}}{\cos\frac{3\pi}{4}}$$

$$= -\frac{2}{3}\left[\frac{1}{-\frac{1}{\sqrt{2}}}\right] = \frac{2\sqrt{2}}{3}$$

Therefore, the equation of the tangent at the point $\left(\frac{1}{\sqrt{2}},0\right)$ is

$$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$y = \frac{2\sqrt{2}}{3}x - \frac{2}{3}$$

$$3y - 2\sqrt{2x} + 2 = 0$$

17.

$$I=\int_{0}^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \qquad \dots\ 1$$

$$I=\int_{0}^{\pi}\frac{\pi-x\ \sin\ \pi-x}{1+\cos^2\ \pi-x}dx$$

$$I=\int_{0}^{\pi}\frac{\pi-x\ \sin x}{1+\cos^2 x}dx$$

$$I=\int_{0}^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx-\int_{0}^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \qquad \dots\ 2$$

Adding (1) and (2), we get:

$$2I=\int_{0}^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx$$

Now, let cos x = t $\Rightarrow$ -sin x dx = dt

When x = π, t = cos π = -1

When x = 0, t = cos 0 = 1

$$2I=\int_{1}^{-1}\pi\ \frac{-dt}{1+t^2}$$

$$2I=-\pi\int_{1}^{-1}\left(\frac{1}{1+t^2}\right)dt$$

$$2I=-\pi\left[\tan^{-1}t\right]_{1}^{-1}$$

$$2I=\pi\left[\tan^{-1}1-\ \tan^{-1}\ -1\ \right]$$

$$2I=\pi\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right)$$

$$2I=\frac{\pi^2}{2}$$

$$\therefore I=\frac{\pi^2}{4}$$

18. $(x^2 - y^2)dx + 2xydy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \qquad ...(1)$$

It is a homogeneous differential equation.

Let $y = vx$ ...(2)

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad ...(3)$$

Substituting (2) and (3) in (1), we get:

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x\ vx}$$

$$v + x\frac{dv}{dx} = \frac{x^2\ v^2 - 1}{2vx^2} = \frac{v^2 - 1}{2v}$$

$$2v^2 + 2vx\frac{dv}{dx} = v^2 - 1$$

$$2vx\frac{dv}{dx} = -v^2 - 1$$

$$\left(\frac{2v}{v^2+1}\right)dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{2v}{v^2+1}dv = -\int \left(\frac{1}{x}\right)dx$$

$$\log\left|v^2 + 1\right| = -\log|x| + \log C$$

$$\log|v^2 + 1| = \log\left|\frac{C}{x}\right|$$

$$v^2 + 1 = \frac{C}{x}$$

$$x\ v^2 + 1\ = C$$

$$x\left[\left(\frac{y}{x}\right)^2 + 1\right] = C$$

$$y^2 + x^2 = Cx \qquad .... 4$$

It is given that when $x = 1$, $y = 1$

$(1)^2 + (1)^2 = C(1)$

$\Rightarrow C = 2$

Thus, the required solution is $y^2 + x^2 = 2x$.

OR

We need to solve the following differential equation

$$\frac{dy}{dx}=\frac{x\ \ 2y-x}{x\ \ 2y+x}$$

$$\frac{dy}{dx}=\frac{2y-x}{2y+x} \qquad \text{... } 1$$

It is a homogeneous differential equation.

Let y = vx ...(2)

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx} \qquad \text{... } 3$$

Substituting (2) and (3) in (1), we get:

$$v+x\frac{dv}{dx}=\frac{x\ \ 2v-1}{x\ \ 2v+1}$$

$$x\frac{dv}{dx}=\frac{2v-1}{2v+1}-v$$

$$x\frac{dv}{dx}=\frac{-2v^2+v-1}{2v+1}$$

$$\left(\frac{2v+1}{-2v^2+v-1}\right)dv=\left(\frac{1}{x}\right)dx$$

$$\left(\frac{2v+1}{2v^2-v+1}\right)dv=\left(-\frac{1}{x}\right)dx$$

Integrating both sides,

$$\int\frac{1}{2}\left(\frac{4v-1+3}{2v^2-v+1}\right)dv=\int\left(-\frac{1}{x}\right)dx$$

$$\int\frac{1}{2}\left(\frac{4v-1}{2v^2-v+1}\right)dv+\int\frac{3}{2}\left(\frac{1}{2v^2-v+1}\right)dv=\int\left(-\frac{1}{x}\right)dx$$

$$\int\frac{1}{2}\left(\frac{4v-1}{2v^2-v+1}\right)dv+\int\frac{3}{4}\left[\frac{1}{v^2-\frac{v}{2}+\frac{1}{2}}\right]dv=\int\left(-\frac{1}{x}\right)dx$$

$$\frac{1}{2}\log\left|2v^2-v+1\right|+\frac{3}{4}\int\left(\frac{1}{v^2-\frac{v}{2}+\frac{1}{16}+\frac{7}{16}}\right)dv=-\log|x|+C$$

$$\frac{1}{2}\log\left|2v^2-v+1\right|+\frac{3}{4}\int\frac{dv}{\left(v-\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}=-\log|x|+C$$

$$\frac{1}{2}\log\left|2v^2-v+1\right|+\frac{3}{4}\times\frac{4}{\sqrt{7}}\tan^{-1}\left(\frac{v-\frac{1}{4}}{\frac{\sqrt{7}}{4}}\right)=-\log|x|+C$$

$$\frac{1}{2}\log\left|2v^2-v+1\right|+\frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{4v-1}{\sqrt{7}}\right)=C-\log|x|$$

Put $v=\frac{y}{x}$

$$\frac{1}{2}\log\left|2\left(\frac{y}{x}\right)^2-\left(\frac{y}{x}\right)+1\right|+\frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{\frac{4y}{x}-1}{\sqrt{7}}\right)=C-\log|x|$$

$$\frac{1}{2}\log\left|\frac{2y^2-xy+x^2}{x^2}\right|+\frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{4y-x}{\sqrt{7}x}\right)=C-\log|x| \quad \ldots 4$$

Now $y=1$ when $x=1$

$$\frac{1}{2}\log\left|\frac{2\ 1^{2}-1\ \ 1+1^{2}}{1^{2}}\right|+\frac{3}{\sqrt{7}}\tan^{-1}\left[\frac{4\ 1\ -1}{\sqrt{7}\ \ 1}\right]=C-\log|1|$$

$$\frac{1}{2}\log 2+\frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right)=C \quad \ldots 5$$

Therefore, form (4) and (5) we get:

$$\frac{1}{2}\log\left|\frac{2y^2 - xy + x^2}{x^2}\right| + \frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{4y - x}{\sqrt{7}x}\right) = \frac{1}{2}\log 2 + \frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) - \log|x|$$

$$\frac{1}{2}\log\left|\frac{2y^2 - xy + x^2}{x^2}\right| - \frac{1}{2}\log 2 + \log|x| = \frac{3}{\sqrt{7}}\left[\tan^{-1}\frac{3}{\sqrt{7}} - \tan^{-1}\left(\frac{4y - x}{\sqrt{7}x}\right)\right]$$

$$\frac{1}{2}\log\left|\frac{2y^2 - xy + x^2}{2x^2}.x^2\right| = \frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{\frac{3x - 4y + x}{\sqrt{7}x}}{1 + \frac{3(4y - x)}{7x}}\right)$$

$$\frac{1}{2}\log\left|\frac{2y^2 - xy + x^2}{2}\right| = \frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{\frac{4(x - y)}{\sqrt{7}x}}{\frac{7x + 12y - 3x}{7x}}\right)$$

$$\frac{1}{2}\log\left|\frac{2y^2 - xy + x^2}{2}\right| = \frac{3}{\sqrt{7}}\tan^{-1}\left\{\frac{\sqrt{7}(x - y)}{(x + 3y)}\right\}$$

19.

$$\cos^2 x\frac{dy}{dx}+y=\tan x$$

$$\frac{dy}{dx}+\sec^2 x.y=\tan x.\sec^2 x$$

It is a linear differential equation of the first order.

Comparing it with $\frac{dy}{dx}+Py=Q,$ we get:

P = $\sec^2 x$ and Q = tan x. $\sec^2 x$

Integration factor = $e^{\int Pdx}=e^{\int \sec^2 xdx}=e^{\tan x}$

The solution of the given linear differential equation is given as:

$$y\left(e^{\tan x}\right)=\int\left(\tan x.\sec^2 x\right).e^{\tan x}\,dx+C$$

Let tan x = t $\Rightarrow$ $\sec^2$ x dx = dt

$$ye^t=\int t.e^t.dt+C$$

$$ye^t=t.e^t-\int 1.e^t dt+C$$

$$ye^t=t.e^t-e^t+C$$

$$ye^t=e^t\left(t-1\right)+C$$

$$ye^{\tan x}=e^{\tan x}\left(\tan x-1\right)+C$$

$$y=\left(\tan x-1\right)+Ce^{-\tan x}$$

20.

Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$$\therefore \vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$\vec{a} \times \vec{c} = \hat{i}\,(z - y) - \hat{j}\,(z - x) + \hat{k}\,(y - x)$ ...(1)

Now, $\vec{a} \times \vec{c} = \vec{b}$

$\vec{b} = \hat{j} - \hat{k}$ ...(2)

Comparing (1) and (2), we get :

$z - y = 0 \Rightarrow z = y$ ...(3)

$z - x = -1$ ...(4)

$y - x = -1$ ...(5)

Also, given that

$\vec{a}.\vec{c} = 3$

$\therefore (\hat{i} + \hat{j} + \hat{k}) . (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$x + y + z = 3$

Using (3), we get, $x + 2y = 3$ ...(6)

Adding (5) and (6), we get

$3y = 2 \Rightarrow y = \dfrac{2}{3}$

$\therefore z = \dfrac{2}{3} \quad \because z = y$

From (6), we have,

$x = 3 - 2y$

$\Rightarrow x = 3 - \dfrac{2 \times 2}{3}$

$\Rightarrow x = \dfrac{9 - 4}{3}$

$\Rightarrow x = \dfrac{5}{3}$

$\therefore \vec{c} = \dfrac{5}{3}\hat{i} + \dfrac{2}{3}\hat{j} + \dfrac{2}{3}\hat{k}$

Thus, the required vector $\vec{c}$ is $\dfrac{5}{3}\hat{i} + \dfrac{2}{3}\hat{j} + \dfrac{2}{3}\hat{k}$.

OR

$\vec{a}+\vec{b}+\vec{c}=0 \Rightarrow \vec{a}+\vec{b}=-\vec{c}$

$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = (-\vec{c}).(-\vec{c})$

$\vec{a}.\vec{a}+2\vec{a}.\vec{b}+\vec{b}.\vec{b}=\vec{c}.\vec{c}$

$|\vec{a}|^2+2|\vec{a}||\vec{b}|\cos\theta+|\vec{b}|^2=|\vec{c}|^2$

$(3)^2+2(3)(5)\cos\theta+(5)^2=(7)^2$

$9+30\cos\theta+25=49$

$30\cos\theta=15 \Rightarrow \cos\theta=\frac{1}{2}$

$\cos\theta=\cos 60^o \Rightarrow \theta=60^o$

Hence proved.

21.

$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

The vector form of this equation is:

$\vec{r}=(3\hat{i}+5\hat{j}+7\hat{k})+\lambda(\hat{i}-2\hat{j}+\hat{k})$

$\vec{r}=\vec{a_1}+\lambda\vec{b}_1 \quad \ldots (1)$

$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$

The vector form of this equation is:

$\vec{r}=(-\hat{i}-\hat{j}-\hat{k})+\lambda(7\hat{i}-6\hat{j}+\hat{k})$

$\vec{r}=\vec{a}_2+\lambda\vec{b}_2$

Therefore, $\vec{a}_1=3\hat{i}+5\hat{j}+7\hat{k}, \vec{b}_1=\hat{i}-2\hat{j}+\hat{k}, \vec{a}_2=-\hat{i}-\hat{j}-\hat{k}$ and $\vec{b}_2=7\hat{i}-6\hat{j}+\hat{k}$

Now, the shortest distance between these two lines is given by:

$$d=\left|\frac{\vec{b}_1\times\vec{b}_2\ .\ \vec{a}_2-\vec{a}_1}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$$

$$\vec{b}_1\times\vec{b}_2=\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 1\\ 7 & -6 & 1\end{bmatrix}$$

$$=\hat{i}\ \ -2+6\ \ -\hat{j}\ \ 1-7\ \ +\hat{k}\ \ -6+14$$

$$=4\hat{i}+6\hat{j}+8\hat{k}$$

$$\Rightarrow\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{4^2+6^2+8^2}=\sqrt{116}$$

$$\vec{a}_2-\vec{a}_1=\ -\hat{i}-\hat{j}-\hat{k}\ -\ 3\hat{i}+5\hat{j}+7\hat{k}$$

$$=-4\hat{i}-6\hat{j}-8\hat{k}$$

$$\therefore d=\left|\frac{4\hat{i}+6\hat{j}+8\hat{k}\ .\ -4\hat{i}-6\hat{j}-8\hat{k}}{\sqrt{116}}\right|=\left|\frac{-16-36-64}{\sqrt{116}}\right|=\left|\frac{-116}{\sqrt{116}}\right|=\sqrt{116}$$

OR

Let $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}=\lambda$

$x=-2+3\lambda, y=-1+2\lambda, z=3+2\lambda$

Therefore, a point on this line is: {(-2+3λ), (-1 + 2λ), (3 + 2λ)}

The distance of the point{(-2+3λ), (-1 + 2λ), (3 + 2λ)} from point (1, 2, 3) = $3\sqrt{2}$

$\therefore\sqrt{(-2+3\lambda-1)^2+(-1+2\lambda-2)^2+(3+2\lambda-3)^2}=3\sqrt{2}$

$\Rightarrow(-3+3\lambda)^2+(-3+2\lambda)^2+(2\lambda)^2=18$

$\Rightarrow 9+9\lambda^2-18\lambda+9+4\lambda^2-12\lambda+4\lambda^2=18$

$17\lambda^2-30\lambda=0$

$\lambda=0, \lambda=\frac{30}{17}$

When $\lambda=\frac{30}{17}$,

$x=-2+3\lambda=-2+3\left(\frac{30}{17}\right)=-2+\frac{90}{17}=\frac{56}{17}$

$y=-1+2\lambda=-1+2\left(\frac{30}{17}\right)=-1+\frac{60}{17}=\frac{43}{17}$

$z=3+2\lambda=3+2\left(\frac{30}{17}\right)=\frac{51+60}{17}=\frac{111}{17}$

Thus, when $\lambda=\frac{30}{17}$, the point is $\left(\frac{56}{17},\frac{43}{17},\frac{111}{17}\right)$ and when λ = 0, the point is (-2, -1, 3).

22. Total number of outcomes = 36

The possible doublets are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), and (6, 6).

Let p be the probability of success, therefore,

$p = \frac{6}{36} = \frac{1}{6}$

So, $q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

Since the dice is thrown 4 times, n=4

Let X denote the number of times of getting doublets in the experiment of throwing two dice simultaneously four times.

Therefore X can take the values 0, 1, 2, 3, or 4.

$$P(X=0) = {}^4C_0 p^0 q^4 = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$$

$$P(X=1) = {}^4C_1 p^1 q^3 = 4\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 = \frac{500}{1296}$$

$$P(X=2) = {}^4C_2 p^2 q^2 = 6\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = \frac{150}{1296}$$

$$P(X=3) = {}^4C_3 p^3 q = 4\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right) = \frac{20}{1296}$$

$$P(X=4) = {}^4C_4 p^4 q^0 = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$$

Thus, the probability distribution is:

X	0	1	2	3	4
P(X)	$\frac{625}{1296}$	$\frac{500}{1296}$	$\frac{150}{1296}$	$\frac{20}{1296}$	$\frac{1}{1296}$

SECTION – C

23.

$$\Delta=\begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \gamma+\alpha & \alpha+\beta\end{vmatrix}$$

Applying $R_3 \to R_3 + R_1$

$$\Delta=\begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma\end{vmatrix}$$

$$= \alpha+\beta+\gamma \begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ 1 & 1 & 1\end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$\Delta= \alpha+\beta+\gamma \begin{vmatrix}\alpha-\beta & \beta-\gamma & \gamma\\ \alpha^2-\beta^2 & \beta^2-\gamma^2 & \gamma^2\\ 0 & 0 & 1\end{vmatrix}$$

$$= \alpha+\beta+\gamma \quad \alpha-\beta \quad \beta-\gamma \begin{vmatrix}1 & 1 & \gamma\\ \alpha+\beta & \beta+\gamma & \gamma^2\\ 0 & 0 & 1\end{vmatrix}$$

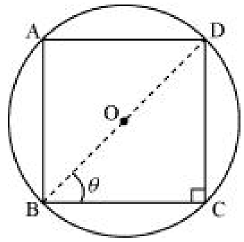
$$= \alpha+\beta+\gamma \quad \alpha-\beta \quad \beta-\gamma \left[1 \;\beta+\gamma \; -1 \;\alpha+\beta\right]$$

$$= \alpha-\beta \quad \beta-\gamma \quad (\alpha+\beta+\gamma)\{\beta+\gamma-\alpha-\beta\}$$

$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

Hence proved.

24. Let a rectangle ABCD be inscribed in a circle with radius *r*.



Let $\angle DBC = \theta$

In right ΔBCD:

$$\frac{BC}{BD} = \cos\theta$$

$$\Rightarrow BC = BD\cos\theta = 2r\cos\theta$$

$$\frac{CD}{BD} = \sin\theta$$

$$\Rightarrow CD = BD\sin\theta = 2r\sin\theta$$

Let A be the area of rectangle ABCD.

$$\therefore A = BC \times CD$$

$$\Rightarrow A = \ 2r\cos\theta \quad 2r\sin\theta \ = 4r^2\sin\theta\cos\theta$$

$$\Rightarrow A = 2r^2\sin 2\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\therefore \frac{dA}{d\theta} = 2.2r^2\cos 2\theta = 4r^2\cos 2\theta$$

Now, $\frac{dA}{d\theta} = 0$

$$\Rightarrow 4r^2\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0$$

$$\Rightarrow \cos 2\theta = \cos\frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$\frac{d^2A}{d\theta^2} = -2.4r^2\sin 2\theta = -8r^2\sin 2\theta$$

$$\therefore \left(\frac{d^2A}{d\theta^2}\right)_{\left(\theta=\frac{\pi}{4}\right)} = -8r^2\sin\left(2.\frac{\pi}{4}\right) = -8r^2.1 = -8r^2 < 0$$

Therefore, by the second derivative test, $\theta = \frac{\pi}{4}$ is the point of local maxima of *A*.

So, the area of rectangle ABCD is the maximum at $\theta = \frac{\pi}{4}$

Now, $\theta = \frac{\pi}{4}$

$$\Rightarrow \frac{CD}{BC} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{CD}{BC} = 1 \Rightarrow CD = BC$$

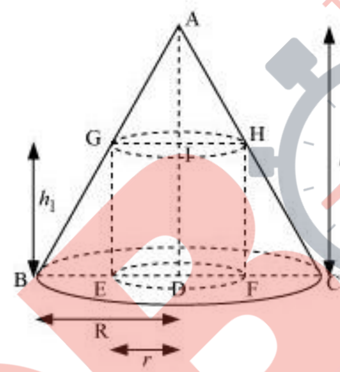
$$\Rightarrow \text{Rectangle ABCD is a square}$$

Hence, the rectangle of the maximum area that can be inscribed in a circle is a square.

OR

Let a cylinder be inscribed in a cone of radius R and height h.

Let the radius of the cylinder be r and its height be h_1.



It can be easily seen that ΔAGI and ΔABD are similar.

$$\therefore \frac{AI}{AD} = \frac{GI}{BD}$$

$$\Rightarrow \frac{h - h_1}{h} = \frac{r}{R}$$

$$\Rightarrow r = \frac{R}{h}(h - h_1)$$

Volume (V) of the cylinder = $\pi r^2 h_1$

$$\Rightarrow V = \pi\frac{R^2}{h^2}(h - h_1)^2 h_1$$

$$\Rightarrow V = \pi\frac{R^2}{h^2}(h^2 + h_1^2 - 2hh_1)h_1$$

$$\Rightarrow \frac{dV}{dh_1} = \pi\frac{R^2}{h^2}\left[(h^2 + h_1^2 - 2hh_1) + h_1(2h_1 - 2h)\right]$$

$$\Rightarrow \frac{dV}{dh_1} = \pi\frac{R^2}{h^2}(h^2 + 3h_1^2 - 4hh_1)$$

Now, $\frac{dV}{dh_1}=0$

$\Rightarrow \frac{\pi R^2}{h^2}\ h^2+3{h_1}^2-4hh_1\ =0$

$\Rightarrow 3h_1^2-4hh_1+h^2=0$

$\Rightarrow 3h_1^2-3hh_1-hh_1+h^2=0$

$\Rightarrow 3h_1\ h_1-h\ -h\ h_1-h\ =0$

$\Rightarrow\ h_1-h\ \ 3h_1-h\ =0$

$\Rightarrow h_1=h, h_1=\frac{h}{3}$

It can be noted that if h_1 = h, then the cylinder cannot be inscribed in the cone.

$\therefore h_1=\frac{h}{3}$

Now, $\frac{d^2V}{dh_1^2}=\frac{\pi R^2}{h^2}\ 0+6h_1-4h\ =\frac{\pi R^2}{h^2}\ 6h_1-4h$

$\therefore \frac{d^2V}{dh_1^2}_{\left(h_1=\frac{h}{3}\right)}=\frac{\pi R^2}{h^2}\left[\frac{6h}{3}-4h\right]=\frac{-2\pi R^2}{h}<0$

Therefore, by the second derivative test, $h_1=\frac{h}{3}$ is the point of local maxima of V.

So, the volume of the cylinder is the maximum when $h_1 = \frac{h}{3}$.

Hence, the height of the cylinder of the maximum volume that can be inscribed in a cone of height *h* is $\frac{1}{3}h$.

25. The respective equations for the parabola and the circle are:

$y^2 = 4x$ ...(1)

$4x^2 + 4y^2 = 9$ ...(2)

or $x^2 + y^2 = \left(\frac{3}{2}\right)^2$

Equation (1) is a parabola with vertex (0, 0) which opens to the right and equation (2) is a circle with centre (0, 0) and radius $\frac{3}{2}$.

From equations (1) and (2), we get:

$4x^2 + 4(4x) = 9$

$4x^2 + 16x - 9 = 0$

$4x^2 + 18x - 2x - 9 = 0$

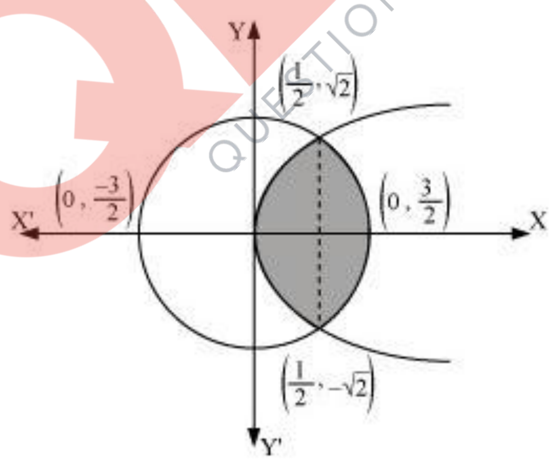
$2x(2x + 9) - 1(2x + 9) = 0$

$(2x + 9)(2x - 1) = 0$

$x = -\frac{9}{2}, \frac{1}{2}$

For $x = -\frac{9}{2}$, $y^2 = 4\left(-\frac{9}{2}\right)$, which is not possible, hence $x = \frac{1}{2}$

Therefore, the given curves intersect at $x = \frac{1}{2}$.



Required area of the region bound by the two curves

$$= 2\int_{0}^{\frac{1}{2}} 2\sqrt{x}\,dx + 2\int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\,dx$$

$$= 4\left[\frac{2}{3}x^{\frac{3}{2}}\right]_{0}^{\frac{1}{2}} + 2\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x}{3}\right)\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{8}{3}\left(\frac{1}{8}\right)^{\frac{1}{2}} + 2\left[0 + \frac{9}{8}\sin^{-1} 1 - \frac{1}{4}\sqrt{2} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{8}{3}\left(\frac{1}{2\sqrt{2}}\right) + \frac{9}{4}\left(\frac{\pi}{2}\right) - \frac{\sqrt{2}}{2} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{2\sqrt{2}}{3} + \frac{9\pi}{8} - \frac{\sqrt{2}}{2} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)$$

26. $I=\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx$

$=\int_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}dx$

$=\int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}dx-\int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}dx$

$=I_1+I_2$

Where $I_1 = \int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}$ dx, which is the integral of an even function

And $I_2 = \int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}$, which is the integral of an odd function, and so $I_2 = 0$

Now,

$I=I_1=\int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}dx$

$=2\int_{0}^{a}\frac{a}{\sqrt{a^2-x^2}}dx$

$=2a\int_{0}^{a}\frac{1}{\sqrt{a^2-x^2}}dx$

$=2a.\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$=2a.\left[\sin^{-1}1-\sin^{-1}0\right]$

$=2a\left(\frac{\pi}{2}\right)$

$=\pi a$

27. The equation of the plane passing through the point (−1, −1, 2) is:

a(x + 1) + b(y + 1) + c (z – 2) = 0 ...(1)

Where a, b and c are the direction ratios of the normal to the plane.

It is given that the plane (1) is perpendicular to the planes.

2x +3y – 3z = 2 and 5x – 4y + z = 6

∴2a + 3b – 3c = 0 ...(2)

5a – 4b + c = 0 ...(3)

Solving equations (2) and (3), we have:

$$\frac{a}{3\times1-\ -4\ \times\ -3}=\frac{b}{-3\ \times5-2\times1}=\frac{c}{2\ -4\ -3\times5}$$

$$\Rightarrow\frac{a}{-9}=\frac{b}{-17}=\frac{c}{-23}$$

So the direction ratios of the normal to the required plane are multiples of 9, 17, and 23.

Thus, the equation of the required plane is:

$$9\ \ x+1\ \ +17\ \ y+1\ \ +23\ \ z-2\ \ =0$$

$$\text{or } 9x+17y+23z=20$$

OR

Equation of the plane passing through the point (3, 4, 1) is:

$$a\ \ x-3\ \ +b\ \ y-4\ \ +c\ \ z-1\ \ =0 \qquad ...\ 1$$

Where *a*, *b*, *c* are the direction ratios of the normal to the plane

It is given that the plane (1) passes through the point (0, 1, 0).

$$\therefore a\ \ -3\ \ +b\ \ -3\ \ +c\ \ -1\ \ =0$$

$$\Rightarrow 3a+3b+c=0 \qquad ...\ 2$$

It is also given that the plane (1) is parallel to the line $\frac{x+3}{2}=\frac{y-3}{7}=\frac{z-2}{5}$.

So, this line is perpendicular to the normal of the plane (1).

$$\therefore 2a+7b+5c=0 \qquad ...\ 3$$

Solving equations (2) and (3), we have:

$$\frac{a}{3\times5-7\times1}=\frac{b}{1\times2-5\times3}=\frac{c}{3\times7-2\times3}$$

$$\Rightarrow\frac{a}{8}=\frac{b}{-13}=\frac{c}{15}$$

So, the direction ratios of the normal to the required plane are multiples of 8, −13, 15.

Therefore, equation (1) becomes:

$$8\ \ x-3\ \ -13\ \ y-4\ \ +15\ \ z-1\ \ =0$$

$\Rightarrow 8x-13y+15z+13=0$, which is the required equation of the plane.

28. Let *x* and *y* respectively be the number of machines A and B, which the factory owner should buy.

Now, according to the given information, the linear programming problem is:

Maximise $Z = 60x + 40y$

Subject to the constraints

$1000x + 1200y \le 9000$

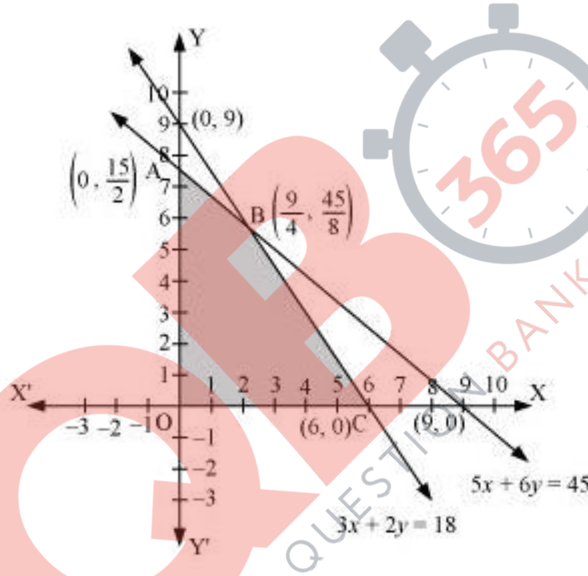
$\Rightarrow 5x + 6y \le 45$ ...(1)

$12x + 8y \le 72$

$\Rightarrow 3x + 2y \le 18$ ...(2)

$x \ge 0, y \ge 0$ ...(3)

The inequalities (1), (2), and (3) can be graphed as:



The shaded portion OABC is the feasible region.

The value of *Z* at the corner points are given in the following table.

Corner point	$Z = 60x + 40y$	
O(0,0)	0	
$A\left(0, \frac{15}{2}\right)$	300	
$B\left(\frac{9}{4}, \frac{45}{8}\right)$	$360 \longrightarrow$	Maximum
C(6,0)	$360 \longrightarrow$	Maximum

The maximum value of *Z* is 360 units, which is attained at $B\left(\frac{9}{4}, \frac{45}{8}\right)$ and C(6,0).

Now, the number of machines cannot be in fraction.

Thus, to maximize the daily output, 6 machines of type A and no machine of type B need to be bought.

29. Let E_1, E_2 and E_3 be the events of a driver being a scooter driver, car driver and truck driver respectively. Let A be the event that the person meets with an accident.

There are 2000 insured scooter drivers, 4000 insured car drivers and 6000 insured truck drivers.

Total number of insured vehicle drivers = 2000 + 4000 + 6000 = 12000

$$\therefore P(E_1) = \frac{2000}{12000} = \frac{1}{6}, P(E_2) = \frac{4000}{12000} = \frac{1}{3}, P(E_3) = \frac{6000}{12000} = \frac{1}{2}$$

Also, we have:

$$P(A|E_1) = 0.01 = \frac{1}{100}$$

$$P(A|E_2) = 0.03 = \frac{3}{100}$$

$$P(A|E_3) = 0.15 = \frac{15}{100}$$

Now, the probability that the insured person who meets with an accident is a scooter driver is $P(E_1|A)$.

Using Bayes' theorem, we obtain:

$$P(E_1|A) = \frac{P(E_1) \times P(A|E_1)}{P(E_1) \times P(A|E_1) + P(E_2) \times P(A|E_2) + P(E_3) \times P(A|E_3)}$$

$$= \frac{\frac{1}{6} \times \frac{1}{100}}{\frac{1}{6} \times \frac{1}{100} + \frac{1}{3} \times \frac{3}{100} + \frac{1}{2} \times \frac{15}{100}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6} + 1 + \frac{15}{2}}$$

$$= \frac{1}{6} \times \frac{6}{52}$$

$$= \frac{1}{52}$$