Very Short Answer Questions (PYQ)

[1 Mark]

Q.1. For what value of '*k*' is the function $f(x) = \begin{cases} \frac{\sin 5x}{3x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ **continuous at *x* = 0?**

Ans.

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} f(h) \qquad = \lim_{h \to 0} \left(\frac{\sin\ 5h}{3h} + \cos\ h\right)$$

$$= \lim_{h \to 0} \frac{\sin\ 5h}{5h} \times \frac{5}{3} + \lim_{h \to 0} \cos\ h \qquad [\because h \to 0 \Rightarrow 5h \to 0]$$

$$= 1 \times \frac{5}{3} + 1$$

$$\lim_{x \to 0^+} f(x) = \frac{8}{3}$$

Also, $f(0) = k$

Since, $f(x)$ is continuous at $x = 0$

$$\Rightarrow \quad \lim_{x \to 0^+} f(x) = f(0) \qquad \Rightarrow \quad \frac{8}{3} = k$$

Q.2. Determine the value of the constant '*k*' so that the function $f(x) = \begin{cases} \frac{}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ **is continuous at *x* = 0.**

Ans.

$\because \quad f(x)$ is continuous at $x = 0$

$$\Rightarrow \quad \lim_{x \to 0^{-}} f(x) = \lim_{x \to 0} f(x) = f(0)$$

Now, $\lim_{x \to 0} f(x) = \lim_{h \to 0} f(0 - h)$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{k(-h)}{|-h|}$$

$$= \lim_{h \to 0} \frac{-}{h} = -k$$

Also, $f(0) = 3$

$$\lim_{x \to 0} f(x) = f(0)$$

$$-k = 3 \qquad \Rightarrow \qquad k = -3$$

Very Short Answer Questions (OIQ)

[1 Mark]

Q.1. Is every differentiable function continuous? Is every continuous function differentiable?

Ans.

Yes, every differentiable function is continuous.

No, some continuous functions may not be differentiable.

Q.2. Find the derivative of $\frac{e^x}{\sin x}$.

Ans.

Let $y = \frac{e^x}{\sin x}$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin x . e^x - e^x \cos x}{\sin^2 x} \qquad \text{[Using quotient rule]}$$

$$= \frac{e^x (\sin x - \cos x)}{\sin^2 x}$$

Q.3. Find $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}}$, where $y = e^{\sin x}$.

Ans.

We have, $y = e^{\sin x}$ $\Rightarrow$ $\frac{dy}{dx} = e^{\sin x} . \cos x$

$\therefore$ $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = e^1 . 0 = 0$

Q.4. If $f(1) = 4$, $f'(1) = 2$, find the value of derivative of log $f(e^x)$ with respect to x at the point $x = 0$.

Ans.

Let $y = \log f(e^x)$

$\therefore$ $\frac{dy}{dx} = \frac{1}{f(e^x)} \times f'(e^x).e^x = \frac{e^x.f'(e^x)}{f(e^x)}$

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{e^0.f'(e^0)}{f.(e^0)} = \frac{1.f'(1)}{f(1)} = \frac{2}{4} = \frac{1}{2}$$

Q.5. Find the derivative of $\log_{10} x$ with respect to x.

Ans.

Let $y = \log_{10} x$

$y = \log_{10} e . \log_e x$

$\therefore$ $\frac{dy}{dx} = \log_{10} e \frac{1}{x} = \frac{\log_{10} e}{x}$ $\left[\because \frac{d}{dx} \log_e x = \frac{1}{x}\right]$

Short Answer Questions (PYQ)

[2 Mark]

Q.1. Find $\frac{dy}{dx}$ at $t=\frac{2\pi}{3}$ **when *x* = 10(*t* – sin *t*) and *y* = 12 (1 – cos *t*).**

Ans.

Given

$$x = 10\ (t - \sin t) \text{ and } y = 12\ (1 - \cos t)$$

$\because \quad x = 10\ (t - \sin t)$

Differentiating w.r.t. t we get

$$\frac{dx}{dt} = 10\ (1 - \cos t)$$

Again $y = 12\ (1 - \cos t)$

$$\Rightarrow \quad \frac{dy}{dt} = 12\ (0 + \sin t) = 12 \sin t$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10\ (1 - \cos t)}$

$$\because \quad \left.\frac{dy}{dx}\right]_{t=\frac{2\pi}{3}} = \frac{12 \sin \frac{2\pi}{3}}{10\left(1 - \cos \frac{2\pi}{3}\right)}$$

$$= \frac{6}{5} \times \frac{\sin\left(\pi - \frac{\pi}{3}\right)}{\left(1 - \cos\left(\pi - \frac{\pi}{3}\right)\right)} = \frac{6}{5} \times \frac{\sin\frac{\pi}{3}}{1 + \cos\frac{\pi}{3}}$$

$$= \frac{6}{5} \times \frac{\frac{\sqrt{3}}{2}}{1 + \frac{1}{2}} = \frac{6\sqrt{3}}{5 \times 3} = \frac{2\sqrt{3}}{5}$$

Q.2. Find $\frac{dy}{dx}$ at $x = 1,\ y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.

Ans.

$\sin^2 y + \cos xy = K$

Differentiating w.r.t. x, we get

$$2\sin y.\cos y\frac{dy}{dx} + (-\sin xy)(x.\frac{dy}{dx} + y) = 0$$

$$\sin 2y.\frac{dy}{dx} - x\sin xy.\frac{dy}{dx} - y\sin xy = 0$$

$$\frac{dy}{dx} = \frac{y\sin xy}{(\sin 2y - x\sin xy)}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{x=1,\ y=\frac{\pi}{4}} = \frac{\frac{\pi}{4}.\sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{4}} = \frac{\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{\pi}{4(\sqrt{2}-1)}$$

Short Answer Questions (OIQ)

[2 Mark]

Q.1. Write the derivative of $f(x) = |x^3|$at $x = 0$.

Ans.

We have,

$$f(x) = |x^3| = \begin{cases} x^3 & ,x \geq 0 \\ -x^3 & ,x < 0 \end{cases}$$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

Now, $f'(0^-) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \lim_{x\to 0} \frac{-x^3 - 0}{x} = -\lim_{x\to 0} x^2 = 0$

and $f'(0^+) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} \lim_{x\to 0^+} \frac{+x^3 - 0}{x} = \lim_{x\to 0^+} x^2 = 0$

Thus, $f'(0) = f'(0-) = f'(0^+) = 0$

Q.2. Find $\frac{dy}{dx}$ if $y + \sin y = \cos x$.

Ans.

Given, $y + \sin y = \cos x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} + \cos y . \frac{dy}{dx} = -\sin x$$

$$\Rightarrow \quad \frac{dy}{dx}(1 + \cos y) = -\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sin x}{1+\cos y} \qquad [\because \; y \neq (2n+1)\pi]$$

Q.3. Differentiate: $2\sqrt{\cot(x^2)}$ **w.r.t.** ***x***

Ans.

Let $y = 2\sqrt{\cot(x^2)}$

Differentiating w.r.t. x both sides, we get

$$\frac{dy}{dx} = 2 \times \frac{1}{2\sqrt{\cot(x^2)}} \times -\text{cosec}^2(x^2) \times 2x$$

$$= \frac{-2x\,\text{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}$$

Q.4. Find $\frac{dy}{dx}$, if $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Ans.

Let $x = \tan\theta$

$\Rightarrow \quad \theta = \tan^{-1} x.$

Now, $\quad y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow \quad y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$\Rightarrow \quad y = \sin^{-1}(\sin 2\theta) \quad \left[\because \ \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$

$\Rightarrow \quad y = 2\theta \quad [\because \sin^{-1}(\sin 2\theta) = 2\theta]$

$\Rightarrow \quad y = 2\tan^{-1} x \quad [\because \ \theta = \tan^{-1} x]$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2.\frac{1}{1+x^2} \quad = \frac{2}{1+x^2}$$

Q.5. Find $\frac{dy}{dx}$, if $y = \frac{\cos x}{\log x}$, $x > 0$.

Ans.

Given, $y = \frac{\cos x}{\log x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\log x \times (-\sin x) - \cos x \times \frac{1}{x}}{(\log x)^2}$$

$$= \frac{-\sin x \log x - \frac{\cos x}{x}}{(\log x)^2} = \frac{-x\sin x.\log x - \cos x}{x(\log x)^2}$$

Q.6. Differentiate w.r.t. x: $e^x + e^{x^2} + \ldots\ldots\ldots + e^{x^5}$

Ans.

Let $y = e^{x} + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{x} + e^{x^2}\,.\,2x + e^{x^3}\,.3x^2 + e^{x^4}\,.4x^3 + e^{x^5}\,.5x^4$$
$$= e^{x} + 2\,xe^{x^2} + 3x^2 e^{x^3} + 4x^3 e^{x^4} + 5x^4 e^{x^5}$$

Q.7. If $y = x^{\sin x}$, $x > 0$, then find $\frac{dy}{dx}$.

Ans.

Given $y = x^{\sin x}$

Taking log of both sides, we get

$\log y = \log x\sin x$

$\Rightarrow \quad \log y = \sin x. \log x$

$\Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = \sin x.\frac{1}{x} + \log x.\cos x$ [Differentiating both sides w.r.t. x]

$\Rightarrow \quad \frac{dy}{dx} = y\left\{\frac{\sin x}{x} + \cos x\log x\right\} = x^{\sin x}\left\{\frac{\sin x}{x} + \cos x\log x\right\}$

Q.8. Find $\frac{dy}{dx}$, if $x = 2at^2$, $y = at^4$.

Ans.

$\because \quad x = 2\,at^2 \quad \Rightarrow \quad \frac{dx}{dt} = 4\,at$

Again, $y = at^4$

$$\frac{dy}{dt} = 4\,at^3$$

Now, $\frac{dy}{dx} = \frac{dy\,/\,dt}{dx\,/\,dt}$

$$= \frac{4\,at^3}{4\,at} = t^2$$

Q.9.

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then prove that [*HOTS*]

(*i*) $C_1 + 2C_2 + ... + nC_n = n \,.\, 2^{n-1}$ (*ii*) $C_1 - 2C_2 + 3C_3 - .... + (-1)^n \, nC_n = 0$

Ans.

We have, $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

Differentiating both sides with respect to x, we have

$$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + nC_nx^{n-1}$$

Putting $x = 1$ and $x = -1$ successively, we have

$$C_1 + 2C_2 + 3C_3 + ... + nC_n = n \,.\, 2^{n-1}$$

and $$C_1 - 2C_2 + 3C_3 - .... + (-1)^n \, nC_n = 0$$

Long Answer Questions-I-A(PYQ)

[4 Marks]

Q.1.

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x} & , \ if\ x<\frac{\pi}{2}\\ p & , \ if\ x=\frac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2} & , \ if\ x>\frac{\pi}{2}\end{cases}$$

is continuous at $x=\frac{\pi}{2}$.

Ans.

We have

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x} & , \ if\ x<\frac{\pi}{2}\\ p & , \ if\ x=\frac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2} & , \ if\ x>\frac{\pi}{2}\end{cases} \qquad \text{is continuous at } x=\frac{\pi}{2}$$

Now, $\lim\limits_{h\to\frac{\pi^+}{2}} f(x)=\lim\limits_{h\to 0} f\left(\frac{\pi}{2}+h\right)$ $\qquad$ $\left[\text{Let } x=\frac{\pi}{2}+h,\ x\to\frac{\pi^+}{2}\Rightarrow h\to 0\right]$

$$=\lim_{h\to 0}\frac{q\left\{1-\sin\left(\frac{\pi}{2}+h\right)\right\}}{\left\{\pi-2\left(\frac{\pi}{2}+h\right)\right\}^2}=\lim_{h\to 0}\frac{q\{1-\cos h\}}{\{\pi-\pi-2h\}^2}=\lim_{h\to 0}\frac{q(1-\cos h)}{4h^2}$$

$$=\lim_{h\to 0}\frac{q.2\sin^2\frac{h}{2}}{4h^2}=\lim_{h\to 0}\frac{q.\sin^2\frac{h}{2}}{2h^2}$$

$$=q.\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\times\frac{1}{8}=\frac{q}{8}$$

Again $\lim\limits_{h\to\frac{\pi^-}{2}} f(x)=\lim\limits_{h\to 0} f\left(\frac{\pi}{2}-h\right)$ $\qquad$ $\left[\text{Let } x=\frac{\pi}{2}-h,\ x\to\frac{\pi^-}{2}\Rightarrow h\to 0\right]$

$$= \lim_{h \to 0} \frac{1 - \sin^3\left(\frac{\pi}{2} - h\right)}{3\cos^2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{1 - \cos^3 h}{3\sin^2 h}$$

$$= \lim_{h \to 0} \frac{(1 - \cos\ h)\ (1 + \cos\ h + \cos^2 h)}{3\ \sin^2 h}$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}.(1+1+1)}{3\sin^2 h}$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}.3}{3\sin^2 h} = \lim_{h \to 0} \frac{2.\sin^2 \frac{h}{2}}{\sin^2 h}$$

$$= \lim_{h \to 0} \frac{2.\frac{\sin^2 \frac{h}{2}}{h^2}}{\frac{\sin^2 h}{h^2}} \qquad \left[\text{Dividing } N^r \text{ and } D^r \text{ by } h^2\right]$$

$$= \lim_{h \to 0} \frac{2.\frac{\sin^2 \frac{h}{2}}{\frac{h^2}{4} \times 4}}{\frac{\sin^2 h}{h^2}} = \frac{1}{2} \frac{\left(\lim_{\frac{h}{2} \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2}{\left(\lim_{h \to 0} \frac{\sin h}{h}\right)^2} = \frac{1}{2}$$

Also $f\ \left(\frac{\pi}{2}\right) = p$

$\because \qquad f(x)$ is continuous at $x = \frac{\pi}{2}$

$\therefore \qquad \lim_{h \to \frac{\pi^+}{2}} f(x) = \lim_{h \to \frac{\pi^-}{2}} f(x) = f\ \left(\frac{\pi}{2}\right)$

$\Rightarrow \qquad \frac{q}{8} = \frac{1}{2} = p$

$\Rightarrow \qquad p = \frac{1}{2}$ and $q = 4$

Q.2. Show that the function *f*(*x*) = 2*x* – |*x*| is continuous but not differentiable at *x* = 0.

Ans.

Here $f(x) = 2x - |x|$

For continuity at x = 0

$$\lim_{h \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \{2h - |h|\} = \lim_{h \to 0} (2h - h)$$

$$= \lim_{h \to 0} h$$

$$= 0 \qquad \ldots(i)$$

$$\lim_{h \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \{2(-h) - |-h|\} = \lim_{h \to 0} \{-2h - h\}$$

$$= \lim_{h \to 0} (-3h)$$

$$= 0 \qquad \ldots(ii)$$

Also, $f(0) = 2 \times 0 - |0| = 0 \qquad \ldots(iii)$

(i),(ii) and (iii) $\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$

For differentiability at x = 0

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(2(-h) - |-h|) - \{2 \times 0 - |0|\}}{-h} = \lim_{h \to 0} \frac{-2h - h - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-3h}{-h} = \lim_{h \to 0} 3$$

$$\text{LHD} = 3 \qquad \ldots(iv)$$

Again RHD $= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{2h - |h| - 2 \times 0 - |0|}{h} = \lim_{h \to 0} \frac{2h - h}{h} = \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1$$

$$\text{RHD} = 1 \qquad \ldots(v)$$

From (iv) and (v)

LHD $\neq$ RHD

i.e., function $f(x) = 2x - |x|$ is not differentiable at $x = 0$

Hence, $f(x)$ is continuous but not differentiable at $x = 0$.

Q.3. Find the value of *k*, for which

$$f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if} -1 \leq x < 0 \\ \frac{2x+1}{x-1}, & \text{if } 0 \leq x < 1 \end{cases}$$

is continuous at *x* = 0.

Ans.

$\because$ $f(x)$ is continuous at $x = 0$

$\Rightarrow$ (LHL of $f(x)$ at $x = 0$) = (RHL of $f(x)$ at $x = 0$) = $f(0)$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \qquad \ldots(i)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(0 - h) \qquad [\text{Let } x = 0 - h,\ x \to 0^- \Rightarrow h \to 0]$$

$$= \lim_{x \to 0} f(-h) = \lim_{h \to 0} \frac{\sqrt{1+k(-h)} - \sqrt{1-k(-h)}}{-h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1-\text{kh}} - \sqrt{1+\text{kh}}}{-h} \times \frac{\sqrt{1-\text{kh}} + \sqrt{1+\text{kh}}}{\sqrt{1-\text{kh}} + \sqrt{1+\text{kh}}}$$

$$= \lim_{h \to 0} \frac{(1-\text{kh}) - (1+\text{kh})}{-h\{\sqrt{1-\text{kh}} + \sqrt{1+\text{kh}}\}} = \lim_{h \to 0} \frac{2k}{\{\sqrt{1-\text{kh}} + \sqrt{1+\text{kh}}\}} = \frac{2k}{2}$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = k \qquad \ldots.(ii)$$

$$\text{Again } \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(0+h) \qquad [\text{Let } x = 0 + h,\ x \to 0^+ \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{2h+1}{h-1} = \frac{1}{-1}$$

$$\Rightarrow \quad \lim_{x \to 0^+} f(x) = -1 \qquad \ldots(iii)$$

$$\text{Also } f(0) = \frac{2 \times 0 + 1}{0 - 1} = -1 \qquad \ldots(iv)$$

$\because$ $f(x)$ is continuous at $x = 0$

$\therefore$ (i), (ii), (iii) and (iv) $\Rightarrow$ $k = -1$.

Q.4. Find the value of '*a*' for which the function *f* defined as

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

is continuous at *x* = 0.

Ans.

$\because$ $f(x)$ is continuous at $x = 0$.

$\Rightarrow$ (LHL of $f(x)$ at $x = 0$) = (RHL of $f(x)$ at $x = 0$) = $f(0)$

$\Rightarrow$ $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$ $\quad \ldots(i)$

Now, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} a \sin\frac{\pi}{2}(x+1)$ $\quad \left[\because f(x) = a \sin\frac{\pi}{2}(x+1), \text{ if } x \leq 0\right]$

$$= \lim_{x \to 0} a \sin\left(\frac{\pi}{2} + \frac{\pi}{2}x\right) = \lim_{x \to 0} a \cos\frac{\pi}{2}x = a.\cos 0 = a \quad \ldots(ii)$$

Again, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$ $\quad \left[\because f(x) = \frac{\tan x - \sin x}{x^3} \text{ if } x\ 0\right]$

$$= \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x \to 0} \frac{\sin x - \sin x . \cos x}{\cos x . x^3} = \lim_{x \to 0} \frac{\sin x(1 - \cos x)}{\cos x . x^3}$$

$$= \lim_{x \to 0} \frac{1}{\cos x} . \lim_{x \to 0} \frac{\sin x}{x} . \frac{2\sin^2 \frac{x}{2}}{\frac{x^2}{4} \times 4} \quad \left[\because 1 - \cos x = 2\sin^2 \frac{x}{2}\right]$$

$$= \frac{1}{1}.1.\frac{1}{2}\lim_{x \to 0}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}.\left(\lim_{\frac{x}{2} \to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}}\right) = \frac{1}{2} \times 1 = \frac{1}{2} \quad \ldots(iii)$$

Also, $f(0) = a\sin\frac{\pi}{2}(0+1) = a\sin\frac{\pi}{2} = a$ $\quad \ldots(iv)$

$\because$ f is continuous at $x = 0$

$\therefore$ $(i), (ii), (iii)$ and (iv) $\Rightarrow$ $a = \frac{1}{2}$

Q.5.

$$\text{If } f(x) = \begin{cases} \frac{\sin (a+1)x + 2 \sin x}{x}, & x < 0 \\ 2, & x = 0 \\ \frac{\sqrt{1+bx} - 1}{x}, & x > 0 \end{cases}$$

is continuous at *x* = 0, then find the values of *a* and *b*.

Ans.

We have

$$f(x)=\begin{cases}\frac{\sin(a+1)x+2\sin x}{x} & , \quad x<0\\ 2 & , \quad x=0\\ \frac{\sqrt{1+bx}-1}{x} & , \quad x>0\end{cases}$$ is continuous at $x = 0$

Since, $f(x)$ is continuous at $x = 0$

$$\Rightarrow \quad \lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x)=f(0) \quad \ldots (i)$$

Now, $\lim_{x\to0^+} f(x)=\lim_{h\to0} f(0+h)$ [Let $x = 0 + h$, h is +ve small quantity $x\to0^+ \Rightarrow h\to0$]

$$=\lim_{h\to0} f(h)$$

$$=\lim_{h\to0}\frac{\sqrt{1+bh}-1}{h}$$

$$=\lim_{h\to0}\frac{\sqrt{1+bh}-1}{h}\times\frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1}$$

$$=\lim_{h\to0}\frac{1+bh-1}{h(\sqrt{1+bh}+1)}$$

$$=\lim_{h\to0}\frac{bh}{h(\sqrt{1+bh}+1)}=\lim_{h\to0}\frac{b}{\sqrt{1+bh}+1}=\frac{b}{2}$$

Again $\lim_{x\to0^-} f(x)=\lim_{h\to0} f(0-h)$ [Let $x = 0 - h$, h is +ve small quantity $x\to0^- \Rightarrow h\to0$]

$$=\lim_{h\to0} f(-h)$$

$$=\lim_{h\to0}\frac{\sin(a+1)(-h)+2\sin(-h)}{-h}$$

$$=\lim_{h\to0}\left[\frac{-\sin(a+1)h-2\sin h}{-h}\right]$$

$$=\lim_{h\to0}\left[\frac{\sin(a+1)h}{h}+\frac{2\sin h}{h}\right]$$

$$=\lim_{h\to0}\frac{\sin(a+1)h}{h}+2\lim_{h\to0}\frac{\sin h}{h}$$

$$=\lim_{h\to0}\frac{\sin(a+1)h}{(a+1)h}\times(a+1)+2\lim_{h\to0}\frac{\sin h}{h}$$

$$=1\times(a+1)+2$$

$$=a+3$$

Also $f(0) = 2$

Now from (i) $\frac{b}{2} = a + 3 = 2$

$\Rightarrow$ $b = 4, \ a = -1$

Q.6. Find the value of *a* and *b* if the function $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$ **is continuous at *x* = 1.**

Ans.

Given function $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$

For continuity at $x = 1$, we have

$f(1) = 11$

LHL $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 5ax - 2b = 5a - 2b$

RHL $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3ax + b = 3a + b$

For $f(x)$ to be continuous at $x = 1$, LHL = RHL = $f(1)$

i.e., $5a - 2b = 3a + b = 11$

On solving, $5a - 2b = 11$ and $3a + b = 11$

We get $a = 3, \quad b = 2.$

Q.7. For what value of *k* is the following function continuous at *x* = 2?

$$f(x) = \begin{cases} 2x + 1, & x < 2 \\ k, & x = 2 \\ 3x - 1, & x > 2 \end{cases}$$

Ans.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2} 2x+1 = 2\times 2+1 = 5 \qquad [\because f(x) = 2x+1, \text{ if } x<2]$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} 3x-1 = 3\times 2-1 = 5 \qquad [\because f(x) = 3x-1, \text{ if } x>2]$$

Since, $f(x)$ is continuous at $x = 2$.

$$\therefore \qquad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \qquad 5 = 5 = k \qquad \Rightarrow \qquad k = 5$$

Q.8. Discuss the continuity of the following function at *x* = 0:

$$f(x) = \begin{cases} \frac{x^4+2x^3+x^2}{\tan^{1} x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Ans.

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) \qquad [\text{Let } x = 0-h, \Rightarrow x \to 0^- \Rightarrow h \to 0]$$

$$= \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} \frac{(-h)^4+2(-h)^3+(-h)^2}{\tan^{-1}(-h)} \qquad [\because -h \neq 0]$$

$$= \lim_{h\to 0} \frac{h^4-2h^3+h^2}{-\tan^{-1} h} \qquad [\because \tan^{-1}(-x) = -\tan^{-1} x]$$

$$= \lim_{h\to 0} \frac{h(h^3-2h^2+h)}{-\tan^{-1}(h)} = \lim_{h\to 0} \frac{h^3-2h^2+h}{-\frac{\tan^{-1} h}{h}}$$

$$= \frac{\lim_{h\to 0} (h^3-2h^2+h)}{\lim_{h\to 0} \frac{-\tan^{-1} h}{h}} = \frac{0}{-1} = 0 \qquad \left[\because \lim_{h\to 0} \frac{\tan^{-1} h}{h} = 1\right]$$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} f(h) \qquad [h \neq 0]$$

$$= \lim_{h \to 0} \frac{h^4 + 2h^3 + h^2}{\tan^{-1} h} = \lim_{h \to 0} \frac{h(h^3 + 2h^2 + h)}{\tan^{-1} h}$$

$$= \lim_{h \to 0} \frac{h^3 + 2h^2 + h}{\frac{\tan^{-1} h}{h}}$$

$$= \frac{\lim_{h \to 0} (h^3 + 2h^2 + h)}{\lim_{h \to 0} \frac{\tan^{-1} h}{h}} = \frac{0}{1} = 0 \qquad \left[\because \lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1\right]$$

$$f(0) = 0 \qquad [\because f(x) = 0 \text{ for } x = 0]$$

i.e., $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$

Hence, $f(x)$ is continuous at $x = 0$.

Q.9. Show that the function '*f*' defined by $f(x) = \begin{cases} 3x - 2, & 0 < x \leq 1 \\ 2x^2 - x, & 1 < x \leq 2 \\ 5x - 4, & x > 2 \end{cases}$ **is continuous at *x* = 2, but not differentiable.**

Ans.

For continuity:

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) \qquad [\text{Let } x = 2-h, \Rightarrow x \to 2^- \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} 2(2-h)^2 - (2-h) = \lim_{h \to 0} 2\{4 + h^2 - 4h\} - (2-h)$$

$$= \lim_{h \to 0} (8 + 2h^2 - 8h - 2 + h) = 6$$

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h) \qquad [\text{Let } x = 2+h, \Rightarrow x \to 2^+ \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} 5(2+h) - 4 = 6$$

$$f(2) = 2(2)^2 - 2 = 6$$

i.e., $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow$ $f(x)$ is continuous at $x = 2$

For Differentiability:

$$\text{LHD (at } x = 2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{2(2-h)^2 - (2-h) - \{2.2^2 - 2\}}{-h} = \lim_{h \to 0} \frac{8 + 2h^2 - 8h - 2 + h - 6}{-h}$$

$$= \lim_{h \to 0} \frac{2h^2 - 7h}{-h} = \lim_{h \to 0} \frac{2h - 7}{-1} = 7$$

$$\text{RHD (at } x = 2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{5(2+h) - 4 - \{2.2^2 - 2\}}{h}$$

$$= \lim_{h \to 0} \frac{10 + 5h - 4 - 6}{h} = \lim_{h \to 0} 5 = 5$$

$\because$ LHD $\neq$ RHD (at $x = 2$)

Hence, $f(x)$ is not differentiable at $x = 2$.

Q.10. Find the relationship between '*a*' and '*b*' so that the function *f* defined by:

$f(x)=\begin{cases} \mathrm{ax}+1, & \text{if } x \leq 3 \\ \mathrm{bx}+3, & \text{if } x>3 \end{cases}$ **is continuous at *x* = 3.**

Ans.

Since, $f(x)$ is continuous at $x = 3$.

$$\Rightarrow \quad \lim_{x \to 3} f(x) = \lim_{x \to 3^+} f(x) = f(3) \qquad \ldots(i)$$

Now, $\lim_{x \to 3} f(x) = \lim_{h \to 0} f(3-h)$ $\quad$ [Let $x = 3 - h$, $x \to 3^- \Rightarrow h \to 0$]

$$= \lim_{h \to 0} a(3-h)+1 \qquad [\because f(x) = ax + 1 \ \forall \ x \leq 3]$$

$$= \lim_{h \to 0} 3a - \mathrm{ah} + 1 = 3a + 1 \qquad \ldots(ii)$$

$\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h)$ $\quad$ [Let $x = 3 + h$, $x \to 3^+ \Rightarrow h \to 0$]

$$= \lim_{h \to 0} b(3+h)+3 = 3b+3 \qquad [\because f(x) = bx + 3 \ \forall \ x > 3] \qquad \ldots(iii)$$

From equations (i), (ii) and (iii)

$3a + 1 = 3b + 3$

$3a - 3b = 2$ $\quad$ or $\quad$ $a - b = \frac{2}{3}$ which is the required relation.

Q.11. Find the value of *k* so that the function *f*, defined by

$f(x)=\begin{cases} \mathrm{kx}+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x>\pi \end{cases}$ **is continuous at *x* = π.**

Ans.

$$\lim_{x \to \pi^-} f(x) = \lim_{h \to 0} f(\pi - h) \qquad [\text{Let } x = \pi - h,\ x \to \pi^- \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} k(\pi - h) + 1 \qquad [\because f(x) = kx + 1 \text{ for } x \le \pi]$$

$$= k\pi + 1$$

$[\text{Let } x = \pi + h,\ x \to \pi^+ \Rightarrow h \to 0]$

$$\lim_{h \to \pi^+} f(x) = \lim_{h \to 0} f(\pi + h) \qquad [\because f(x) = \cos x \text{ for } x > \pi]$$

$$= \lim_{h \to 0} \cos(\pi + h)$$

$$= \lim_{h \to 0} -\cos h = -1$$

Also $f(\pi) = k\pi + 1$

Since, $f(x)$ is continuous at $x = \pi$.

$$\Rightarrow \quad \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi) \Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow \quad k\pi = -2 \qquad \Rightarrow \quad k = -\frac{2}{\pi}$$

Q.12. Show that the function $f(x) = |x - 3|$, $x \in$ |R, is continuous but not differentiable at $x = 3$.

Ans.

Here, $f(x) = |x - 3|$ $\Rightarrow$ $f(x) = \begin{cases} -(x-3) & , x < 3 \\ 0 & , x = 3 \\ (x-3) & , x > 3 \end{cases}$

For Continuity:

Now, $\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h)$ [Let $x = 3 + h$ and $x \to 3^+ \Rightarrow h \to 0$]

$$= \lim_{h \to 0} (3 + h - 3) = \lim_{h \to 0} h = 0$$

$$\lim_{x \to 3^+} f(x) = 0 \quad ...(i)$$

$\lim_{x \to 3} f(x) = \lim_{h \to 0} f(3 - h)$ [Let $x = 3 - h$ and $x \to 3^- \Rightarrow h \to 0$]

$$= \lim_{h \to 0} -(3 - h - 3) = \lim_{h \to 0} h = 0$$

$$\lim_{x \to 3^+} f(x) = 0 \quad ...(ii)$$

Also, $f(3) = 0$ ...(*iii*)

From equation (*i*), (*ii*) and (*iii*)

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3)$$

Hence, $f(x)$ is continuous at $x = 3$

For Differentiability:

$$\text{RHD} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h-3) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1 \quad ...(iv)$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(3-h) - f(3)}{-h} = \lim_{h \to 0} \frac{-(3-h-3) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} (-1) = -1 \quad ...(v)$$

Equation (iv) and (v)

$\Rightarrow$ RHD $\neq$ LHD at $x = 3$.

Hence, $f(x)$ is not differentiable at $x = 3$.

Therefore, $f(x) = |x - 3|, x \in \mathbb{R}$ is continuous but not differentiable at $x = 3$.

Q.13. Discuss the continuity and differentiability of the function

***f*(*x*) = |*x*| + |*x* – 1| in the interval (–1, 2).**

Ans.

Given function is

$$f(x) = |x| + |x - 1|$$

Function is also written as

$$f(x) = \begin{cases} -x-(x-1), & if -1 < x < 0 \\ 1, & if\ 0 \le x < 1 \\ x+(x-1), & if\ x \ge 1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -2x+1, & \text{if } x < 0 \\ 1, & \text{if } 0 \le x < 1 \\ 2x-1, & \text{if } x \ge 1 \end{cases}$$

Obviously, in given function we need to discuss the continuity and differentiability of the function $f(x)$ at $x = 0$ or 1 only.

For continuity at $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) \qquad [\text{Let } x = 0 + h \text{ and } x \to 0^+ \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} 1 \qquad [\because h \text{ is very small positive quantity}]$$

$$= 1 \qquad \dots(i)$$

$$\lim_{x \to 0} f(x) = \lim_{h \to 0} f(0-h) \qquad [\text{Let } x = 0 - h \text{ and } x \to 0^- \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \{-2(-h)+1\} = \lim_{h \to 0} (2h+1)$$

$$\lim_{x \to 0} f(x) = 1 \qquad \ldots(ii)$$

Also, $f(0) = 1$ ...(*iii*)

(*i*), (*ii*) and (*iii*) ⇒ $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$

For differentiability at $x = 0$

$$\text{RHD} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \qquad [\because h \text{ is very small positive quantity } \Rightarrow 0 < h < 1]$$

$$= \lim_{h \to 0} \frac{1-1}{h} = \lim_{h \to 0} \frac{0}{h} \qquad [\because |h| = h, |0| = 0]$$

$$= \lim_{h \to 0} 0$$

$$\text{RHD} = 1 \qquad \ldots(iv)$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-2(-h)+1-1}{-h} = \lim_{h \to 0} \frac{2h}{-h}$$

$$= \lim_{h \to 0} (-2)$$

$$\text{LHD} = -2 \qquad \ldots(v)$$

(*iv*) and (*v*) ⇒ RHD ≠ LHD at $x = 0$.

Hence, $f(x)$ is not differentiable at $x = 0$ but continuous at $x = 0$.

Similarly, we can prove $f(x)$ is not differentiable at $x = 1$ but continuous at $x = 1$

(Do yourself)

Q.14. Show that the function $f(x) = |x - 1| + |x + 1|$, for all $x \in R$, is not differentiable at the points $x = -1$ and $x = 1$.

Ans.

Here, given function is

$$f(x) = |x - 1| + |x + 1|$$

$$\Rightarrow \quad f(x) = \begin{cases} -(x-1)-(x+1), & x < -1 \\ 2, & x = -1 \\ -(x-1)+(x+1), & -1 < x < 1 \\ 2 & x = 1 \\ (x-1)+(x+1) & x > 1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -2x, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ 2, & \text{if } -1 < x < 1 \\ 2, & \text{if } x = 1 \\ 2x, & \text{if } x > 1 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} -2x & \text{if } x < -1 \\ 2 & \text{if } -1 \le x \le 1 \\ 2x & \text{if } x > 1 \end{cases}$$

For $x = -1$

$$\text{RHD} = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{2-2}{h} = \lim_{h \to 0} 0 = 0$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(-1-h) - f(-1)}{-h}$$

$$= \lim_{h \to 0} \frac{-2(-1-h) - 2}{-h} = \lim_{h \to 0} \frac{2+2h-2}{-h}$$

$$= \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} -2 = -2$$

i.e., RHD $\neq$ LHD.

Hence, $f(x)$ is not differentiable at $x = -1$

For $x = 1$

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{2(1+h) - 2}{h} = \lim_{h \to 0} \frac{2+2h - 2}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(1 - h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{2 - 2}{-h} = \lim_{h \to 0} \frac{0}{-h}$$

$$= \lim_{h \to 0} 0 = 0$$

RHD ≠ LHD.

Hence, $f(x)$ not differentiable at $x = 1$.

Long Answer Questions-I-A(OIQ)

[4 Marks]

Q.1. For what value of *k*, the following function is continuous at *x* = 0?

$$f(x) = \begin{cases} \frac{1 - \cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

Ans.

Given function, $f(x) = \begin{cases} \frac{1 - \cos 4x}{8x^2} & , \quad x \neq 0 \\ k & , \quad x = 0 \end{cases}$

At $x = 0$, we have $f(0) = k$

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1 - \cos 4x}{8x^2}$$

$$= \lim_{x \to 0^-} \frac{2 \sin^2 2x}{8x^2} = \lim_{x \to 0^-} 2 \times \left(\frac{\sin 2x}{2x}\right)^2 \times \frac{1}{2} = 1$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1 - \cos 4x}{8x^2} = \lim_{x \to 0^+} 2 \times \left(\frac{\sin 2x}{2x}\right)^2 \times \frac{1}{2} = 1$$

For $f(x)$ to be continuous at $x = 0$

$\text{LHL} = \text{RHL} = f(0)$

$\Rightarrow \quad 1 = 1 = k \quad \therefore \quad k = 1$

Q.2. Examine the continuity of the following function:

$$f(x) = \begin{cases} \frac{x}{2|x|}, x \neq 0 \\ \frac{1}{2}, \, x = 0 \end{cases} \quad \text{at } x = 0.$$

Ans.

Given, $f(x)=\begin{cases} \frac{x}{2|x|}, & x \neq 0 \\ \frac{1}{2}, & x=0 \end{cases}$

For continuity at $x = 0$, we have

$$f(0)=\frac{1}{2}$$

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{2|x|} = \lim_{x\to 0^-} \frac{x}{-2x} = -\frac{1}{2}$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{2|x|} = \lim_{x\to 0^+} \frac{x}{2x} = \frac{1}{2}$$

Hence, LHL $\neq$ RHL

So, $f(x)$ is discontinuous at $x = 0$.

Q.3. If the function *f*, as defined below is continuous at *x* = 0, find the values of *a*, *b* and *c*.

$$f(x)=\begin{cases} \frac{\sin\,(a+1)\,x+\sin x}{x}, & x<0 \\ c, & x=0 \\ \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, & x>0 \end{cases}$$

Ans.

Since $f(x)$ is continuous at $x = 0$ $\quad \therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Now, $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$ $\quad$ [Let $x = 0 - h$, $x \to 0^- \Rightarrow h \to 0$]

$$= \lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{\sin(a+1)(-h)+\sin(-h)}{-h}$$

$$= \lim_{h\to 0} \frac{\sin(a+1)h+\sin h}{h} = \lim_{h\to 0} \frac{\sin(a+1)h}{h} + \lim_{x\to 0} \frac{\sin h}{h}$$

$$= \lim_{(a+1)h\to 0} \frac{\sin(a+1)h}{(a+1)h} \times (a+1) + 1 \qquad \left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right]$$

$$= 1.(a+1) + 1 \qquad \left[\because \lim_{(a+1)h\to 0} \frac{\sin(a+1)h}{(a+1)h} = 1\right]$$

$$\Rightarrow \quad \lim_{x\to 0^-} f(x) = a + 2 \qquad ...(i)$$

Again, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$ $\quad$ [Let $x = 0 + h$, $x \to 0^+ \Rightarrow h \to 0$]

$$= \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{\sqrt{h+bh^2} - \sqrt{h}}{bh^{3/2}} = \lim_{h\to 0} \frac{\sqrt{h}(\sqrt{1+bh} - 1)}{bh^{3/2}}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+bh} - 1}{bh} \times \frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1} = \lim_{h\to 0} \frac{1+bh-1}{bh\ (\sqrt{1+bh}+1)}$$

$$= = \lim_{h\to 0} \frac{1}{\sqrt{1+bh}+1} = \frac{1}{2}$$

$$\Rightarrow \quad \lim_{x\to 0^+} f(x) = \frac{1}{2} \qquad ...(ii)$$

Also, $\quad f(0) = c \qquad ...(iii)$

Hence, (i), (ii) and $(iii) \Rightarrow \quad a + 2 = \frac{1}{2} = c \quad \Rightarrow \quad a = -\frac{3}{2}$, $c = \frac{1}{2}$ and continuity of f does not depend on the value of b

Long Answer Questions-I-B (PYQ)

[4 Mark]

Q.1. If $y^x = e^{y-x}$, then prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

Ans.

Given, $y^x = e^{y-x}$

Taking logarithm both sides, we get

$$\log y^x = \log e^{y-x}$$

$\Rightarrow \quad x.\log y = (y-x)\,.\log e \quad \Rightarrow \quad x\,.\log y = (y-x) \qquad [\because \log (m)^n = n \log m]$

$\Rightarrow \quad x\,(1+\log y) = y \qquad \Rightarrow \quad x = \frac{y}{1+\log y}$

Differentiating both sides with respect to y, we get

$$\frac{dx}{dy} = \frac{(1+\log y).1 - y.\left(0+\frac{1}{y}\right)}{(1+\log y)^2}$$

$$= \frac{1+\log y - 1}{(1+\log y)^2} = \frac{\log y}{(1+\log y)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$$

$$\begin{bmatrix} \text{Note} : (i)\ \log_e mn = \log_e m + \log_e n \\ (ii)\ \log_e \frac{m}{n} = \log_e m - \log_e n \\ (iii)\ \log_e m^n = n\log_e m \\ (iv)\ \log e = 1 \end{bmatrix}$$

Q.2. If $x^y = e^{x-y}$, then show that $\frac{dy}{dx} = \frac{\log x}{\{\log (xe)\}^2}$.

Ans.

Given, $x^y = e^{x-y}$

Taking log both sides, we get

$$\Rightarrow \quad \log x^y = \log e^{x-y} \quad \Rightarrow \quad y.\log x = (x-y) . \log e \quad [\because \log e = 1]$$

$$\Rightarrow \quad y. \log x = (x-y) \quad \Rightarrow \quad y\log x + y = x$$

$$\Rightarrow \quad y = \frac{x}{1+\log x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{(1+\log x).1 - x.\left(0+\frac{1}{x}\right)}{(1+\log x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(\log e+\log x)^2} \quad [\because 1 = \log e]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{(\log ex)^2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{\{\log (ex)\}^2}$$

Q.3. **Prove that :** $\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] = \sqrt{a^2 - x^2}$

Ans.

$$\text{LHS} = \frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2 - x^2}\right) + \frac{d}{dx}\left(\frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right) = \frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2 - x^2}\right) + \frac{d}{dx}\left(\frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right)$$

$$= \frac{1}{2}\left\{x.\frac{1}{2\sqrt{a^2 - x^2}} \times -2x + \sqrt{a^2 - x^2}\right\} + \frac{a^2}{2}.\frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \times \frac{1}{a} \quad \left[\begin{array}{l}\text{Apply product rule} \\ \text{and inverse formula}\end{array}\right]$$

$$= \frac{-x^2}{2\sqrt{a^2 - x^2}} + \frac{\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2\sqrt{a^2 - x^2}}$$

$$= \frac{-x^2 + a^2 - x^2 + a^2}{2\sqrt{a^2 - x^2}} = \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} = \sqrt{a^2 - x^2} = \text{RHS}$$

Q.4. **If $(\cos x)^y = (\cos y)^x$, then find $\frac{dy}{dx}$.**

Ans.

Given, $(\cos x)^y = (\cos y)^x$

Taking logrithm both sides, we get

$$\log (\cos x)^y = \log (\cos y)^x$$

$\Rightarrow \quad y \cdot \log (\cos x) = x \cdot \log (\cos y) \qquad [\because \log m^n = n \log m]$

Differentiating both sides, we get

$$y \cdot \frac{1}{\cos x}(-\sin x) + \log(\cos x) \cdot \frac{dy}{dx} = x \cdot \frac{1}{\cos y} \cdot (-\sin y) \cdot \frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \quad -\frac{y \sin x}{\cos x} + \log(\cos x) \cdot \frac{dy}{dx} = -\frac{x \sin y}{\cos y} \cdot \frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \quad \log(\cos x) \cdot \frac{dy}{dx} + \frac{x \sin y}{\cos y} \cdot \frac{dy}{dx} = \log(\cos y) + \frac{y \sin x}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\log(\cos x) + \frac{x \sin y}{\cos y}\right] = \log(\cos y) + \frac{y \sin x}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log(\cos y) + \frac{y \sin x}{\cos x}}{\log(\cos x) + \frac{x \sin y}{\cos y}} = \frac{\log(\cos y) + y \tan x}{\log(\cos x) + x \tan y}$$

Q.5.

Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^\theta(\sin\theta - \cos\theta)$ and $y = ae^\theta(\sin\theta + \cos\theta)$.

Ans.

Given, $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$

$$x = ae^{\theta}(\sin\theta - \cos\theta)$$

Differentiating with respect to q, we get

$$\frac{dx}{d\theta} = ae^{\theta}(\cos\theta + \sin\theta) + a(\sin\theta - \cos\theta).e^{\theta} = ae^{\theta}(\cos\theta + \sin\theta + \sin\theta - \cos\theta$$

$$= 2\,ae^{\theta}\sin\theta \qquad \ldots (i)$$

Again, $\because y = ae^{\theta}(\sin\theta + \cos\theta)$

Differentiating with respect to θ, we get

$$\frac{dy}{d\theta} = ae^{\theta}(\cos\theta - \sin\theta) + a(\sin\theta + \cos\theta).e^{\theta} = ae^{\theta}(\cos\theta - \sin\theta + \sin\theta + \cos\theta$$

$$= 2\,ae^{\theta}.\cos\theta \qquad \ldots (ii)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\,ae^{\theta}.\cos\theta}{2\,ae^{\theta}.\sin\theta} \qquad [\text{From } (i) \text{ and } (ii)]$$

$$\Rightarrow \quad \frac{dy}{dx} = \cot\theta \quad \Rightarrow \quad \left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

Q.6. **If $\sin y = x\sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.**

Ans.

Here, $\sin y = x\sin(a+y) \quad \Rightarrow \quad \frac{\sin y}{\sin(a+y)} = x$

$$\Rightarrow \quad \frac{\sin(a+y).\cos y.\frac{dy}{dx} - \sin y.\cos(a+y).\frac{dy}{dx}}{\sin^2(a+y)} = 1$$

$$\Rightarrow \quad \frac{dy}{dx}\{\sin(a+y).\cos y - \sin y.\cos(a+y)\} = \sin^2(a+y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y-y)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Q.7. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ with respect to x.

Ans.

Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Put $x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1} x$

Now, $y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\left[\begin{array}{ll} \because & -\infty\, x\, \infty \\ \Rightarrow & \tan\left(-\frac{\pi}{2}\right)\tan\theta\tan\left(\frac{\pi}{2}\right) \\ \Rightarrow & -\frac{\pi}{2}\theta\frac{\pi}{2} \\ \Rightarrow & -\frac{\pi}{4}\frac{\theta}{2}\frac{\pi}{4} \\ \Rightarrow & \frac{\theta}{2} \in \left(-\frac{\pi}{4},\frac{\pi}{4}\right) \subset \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \end{array}\right]$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$\Rightarrow \quad y = \frac{1}{2}\tan^{-1}x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

Q.8. Differentiate the following with respect to *x*:

$(\sin x)^x + (\cos x)^{\sin x}$

Ans.

Let $u = (\sin x)^{x}$ and $v = (\cos x)^{\sin x}$

$\therefore$ Given differential equation becomes $y = u + v$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(i)$$

Now, $u = (\sin x)^{x}$

Taking log on both sides, we get

$$\log u = x \log \sin x$$

Differentiating with respect to x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = x \frac{1}{\sin x} . \cos x + \log \sin x \quad \Rightarrow \quad \frac{du}{dx} = u(x \cot x + \log \sin x)$$

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^{x} \{x \cot x + \log \sin x\} \qquad \ldots(ii)$$

Again $v = (\cos x)^{\sin x}$

Taking log on both sides, we get

$$\log v = \sin x . \log \cos x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = \sin x . \frac{1}{\cos x} (- \sin x) + \log (\cos x) . \cos x$$

$$\Rightarrow \quad \frac{dv}{dx} = v \left\{ - \frac{\sin^2 x}{\cos x} + \cos x . \log \cos x \right\} = (\cos x)^{\sin x} \left\{ \cos x . \log (\cos x) - \frac{\sin^2 x}{\cos x} \right\}$$

$$\Rightarrow \quad \frac{dv}{dx} = (\cos x)^{1 + \sin x} \{ \log (\cos x) - \tan^2 x\} \qquad \ldots(iii)$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\sin x)^{x} \{x \cot x + \log \sin x\} + (\cos x)^{1 + \sin x} \{ \log (\cos x) - \tan^2 x\}$$

Q.9. If cos *y* = *x* cos (*a* + *y*), with cos *a* ≠ ±1, then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Hence show that $\sin a \frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$

Ans.

To prove $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$ (Refer Q. 30 Page-208)

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{\sin a}\left\{-2\cos(a+y).\sin(a+y).\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad \sin a\frac{d^2y}{dx^2} = -\sin 2(a+y).\frac{dy}{dx}$$

$$\Rightarrow \quad \sin a\frac{d^2y}{dx^2} + \sin 2(a+y).\frac{dy}{dx} = 0$$

Q.10. Differentiate the following with respect to: $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Ans.

Let $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=y$

$$y=\tan^{-1}\left(\frac{1-\frac{\sqrt{1-x}}{\sqrt{1+x}}}{1+\frac{\sqrt{1-x}}{\sqrt{1+x}}}\right)$$

$$\Rightarrow \quad y=\tan^{-1} 1-\tan^{-1}\left(\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)$$

$$\frac{dy}{dx}=0-\frac{1}{1+\left(\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)^2}\cdot\frac{d}{dx}\left(\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)$$

$$=-\frac{1+x}{2}\left\{\frac{\frac{-1}{2\sqrt{1-x}}\sqrt{1+x}-\frac{1}{2\sqrt{1+x}}\sqrt{1-x}}{1+x}\right\}$$

$$=\frac{1+x}{4}\left\{\frac{\frac{\sqrt{1+x}\times\sqrt{1+x}}{\sqrt{1-x}\times\sqrt{1+x}}+\frac{\sqrt{1-x}\times\sqrt{1-x}}{\sqrt{1+x}\times\sqrt{1-x}}}{1+x}\right\}$$

$$=\frac{1}{4}\cdot\frac{2}{\sqrt{1-x^2}}=\frac{1}{2\sqrt{1-x^2}}$$

Q.11. **If $y=\log\,\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$, then show that $\frac{dy}{dx}-\sec\,x=0$.**

Ans.

Given, $y=\log\,\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\frac{d}{dx}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=\cot\left(\frac{\pi}{4}+\frac{x}{2}\right).\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right).\frac{1}{2}$$

$$=\frac{\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\frac{1}{2}\frac{1}{\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\frac{1}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right).\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$

$$=\frac{1}{\sin 2\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\frac{1}{\sin\left(\frac{\pi}{2}+x\right)}=\frac{1}{\cos x}=\sec x$$

$$\Rightarrow \quad \frac{dy}{dx}-\sec x=0$$

Hence proved.

Q.13. Differentiate the following function with respect to x: $(\log x)^x + x^{\log x}$.

Ans.

Let $y = (\log x)^x + x^{\log x}$

$\Rightarrow \quad y = u + v, \quad \text{where } u = (\log x)^x, \quad v = x^{\log x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(i)$$

Now, $\quad u = (\log x)^x$

Taking logarithm on both sides, we get

$$\log u = x \cdot \log(\log x)$$

Differentiating both sides with respect to x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x) \qquad \Rightarrow \qquad \frac{du}{dx} = u\left\{\frac{1}{\log x} + \log(\log x)\right\}$$

$$\Rightarrow \quad \frac{du}{dx} = (\log x)^x \left\{\frac{1}{\log x} + \log(\log x)\right\} \qquad \ldots(ii)$$

Again $\quad v = x^{\log x}$

Taking logarithm of both sides, we get

$$\log v = \log x^{\log x}$$

$$\Rightarrow \quad \log v = \log x \cdot \log x \qquad \Rightarrow \qquad \log v = (\log x)^2$$

Differentiating both sides with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = 2\log x \cdot \frac{1}{x}$$

$$\Rightarrow \quad \frac{dv}{dx} = 2x^{\log x} \cdot \frac{\log x}{x} \qquad \ldots(iii)$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\log x)^x \left\{\frac{1}{\log x} + \log(\log x)\right\} + 2\frac{\log x \cdot x^{\log x}}{x}$$

Q.14.

If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then show that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

Ans.

Given, $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$

Putting $x=\sin\alpha \Rightarrow \alpha=\sin^{-1}x$ and $y=\sin\beta \Rightarrow \beta=\sin^{-1}y$, we get

$$\sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta)$$

$\Rightarrow \quad \cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$

$\Rightarrow \quad 2\cos\frac{(\alpha+\beta)}{2}\cos\left(\frac{\alpha-\beta}{2}\right)=a.2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$\Rightarrow \quad \cot\left(\frac{\alpha-\beta}{2}\right)=a \quad \Rightarrow \quad \frac{\alpha-\beta}{2}=\cot^{-1}a \quad \Rightarrow \quad \alpha-\beta=2\cot^{-1}a$

$\Rightarrow \quad \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

Q.15. **Differentiate** $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ **with respect to** $\sin^{-1}(2x\sqrt{1-x^2})$.

Ans.

Let $u=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and $v=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

We have to determine $\frac{du}{dv}$

Put $x=\sin\theta \quad \Rightarrow \quad \theta=\sin^{-1}x$

Now, $u=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right) \quad \Rightarrow \quad u=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \quad \Rightarrow \quad u=\theta$

$\Rightarrow \quad u=\sin^{-1}x \quad \Rightarrow \quad \frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$

Again, $v=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$\Rightarrow \quad v=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)=\sin^{-1}\left(2\sin\theta\cos\theta\right)$

$\Rightarrow \quad v=\sin^{-1}(\sin 2\theta) \quad \Rightarrow \quad v=2\theta$

$\Rightarrow \quad v=2\sin^{-1}x \quad \Rightarrow \quad \frac{dv}{dx}=\frac{2}{\sqrt{1-x^2}}$

$$\begin{bmatrix} \because & -\frac{1}{\sqrt{2}}x\frac{1}{\sqrt{2}} \\ \Rightarrow & \sin\left(-\frac{\pi}{4}\right)\sin\theta\sin\left(\frac{\pi}{4}\right) \\ \Rightarrow & -\frac{\pi}{4}\theta\frac{\pi}{4} \\ \Rightarrow & -\frac{\pi}{2}2\theta\frac{\pi}{2} \\ \Rightarrow & 2\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \end{bmatrix}$$

$\therefore \quad \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}}=\frac{1}{2}$

[**Note:** Here the range of x is taken as $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$]

Q.16.

If $x = \cos t(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, then find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

Ans.

Given, $x = \cos t(3 - 2\cos^2 t)$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = \cos t\{0 + 4\cos t.\sin t\} + (3 - 2\cos^2 t).(-\sin t)$$

$$= 4\sin t.\cos^2 t - 3\sin t + 2\cos^2 t.\sin t$$

$$= 6\sin t\cos^2 t - 3\sin t = 3\sin t\,(2\cos^2 t - 1) = 3\sin t.\cos 2t$$

Again, $\because\ y = \sin t\,(3 - 2\sin^2 t)$

Differentiating both sides with respect to t, we get

$$\frac{dy}{dt} = \sin t.\{0 - 4\sin t\cos t\} + (3 - 2\sin^2 t\}.\cos t$$

$$= -4\sin^2 t.\cos t + 3\cos t - 2\sin^2 t.\cos t = 3\cos t - 6\sin^2 t.\cos t$$

$$= 3\cos t\,(1 - 2\sin^2 t) = 3\cos t.\cos 2t$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t.\cos 2t}{3\sin t.\cos 2t}$

$$\frac{dy}{dx} = \cot t$$

$\therefore\quad \left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$

Q.17.

Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$.

Ans.

Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

We have to determine $\frac{du}{dv}$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

Now, $u = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$

$\Rightarrow u = \tan^{-1}(\cot\theta) \Rightarrow u = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right]$

$\Rightarrow u = \frac{\pi}{2} - \theta \Rightarrow u = \frac{\pi}{2} - \sin^{-1} x$

$\Rightarrow \frac{du}{dx} = 0 - \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$

Again, $v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

$\because x = \sin\theta$

$\therefore v = \cos^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow v = \cos^{-1}(2\sin\theta \,.\, \cos\theta)$

$\Rightarrow v = \cos^{-1}(\sin 2\theta)$

$\Rightarrow v = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$

$\Rightarrow v = \frac{\pi}{2} - 2\theta \Rightarrow v = \frac{\pi}{2} - 2\sin^{-1} x$

$\Rightarrow \quad \frac{dv}{dx} = 0 - \frac{2}{\sqrt{1-x^2}} \qquad \Rightarrow \qquad \frac{dv}{dx} = - \frac{2}{\sqrt{1-x^2}}$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

$$\left[\begin{array}{ll} \because & -\frac{1}{\sqrt{2}} x \frac{1}{\sqrt{2}} \\ \Rightarrow & \sin\left(-\frac{\pi}{4}\right) \sin\theta \sin\left(\frac{\pi}{4}\right) \\ \Rightarrow & -\frac{\pi}{4} \theta \frac{\pi}{4} \\ \Rightarrow & -\frac{\pi}{2} 2\theta \frac{\pi}{2} \\ \Rightarrow & \frac{\pi}{2} - 2\theta - \frac{\pi}{2} \\ \Rightarrow & \pi\left(\frac{\pi}{2} - 2\theta\right) 0 \\ \Rightarrow & \left(\frac{\pi}{2} - 2\theta\right) \in (0, \pi) \subset [0, \pi] \end{array}\right]$$

[**Note:** Here the range of x is taken as $-\frac{1}{\sqrt{2}} < x > \frac{1}{\sqrt{2}}$]

Q.18. **Find** $\frac{dy}{dx}$, if $(x^2 + y^2)^2 = xy$.

Ans.

Given, equation is $(x^2 + y^2)^2 = xy$.

Differentiating with respect to x, we get

$$2(x^2 + y^2).\left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad 4x(x^2 + y^2) + 4y.(x^2 + y^2).\frac{dy}{dx} = x.\frac{dy}{dx} + y$$

$$\Rightarrow \quad \{4y(x^2 + y^2) - x\}\frac{dy}{dx} = y - 4x(x^2 + y^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - 4x(x^2+y^2)}{4y(x^2+y^2) - x}$$

Q.19. Differentiate the following function with respect to *x*:

$y = (\sin x)^x + \sin^{-1} \sqrt{x}$

Ans.

Given, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$$y = u + v, \quad \text{where } u = (\sin x)^x, \; v = \sin^{-1}\sqrt{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(i)$$

Now, $u = (\sin x)^x$

Taking log both sides, we get

$$\log u = \log(\sin x)^x \quad \Rightarrow \quad \log u = x \,.\, \log(\sin x)$$

Differentiating both sides with respect to x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x.\frac{1}{\sin x}\cos x + \log\ \sin x$$

$$\Rightarrow \quad \frac{du}{dx} = u\{x\cot x + \log\ \sin x\}$$

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^x\{x\cot x + \log\ \sin x\} \qquad \ldots(ii)$$

Also, $v = \sin^{-1}\sqrt{x}$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1 - (\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1 - x)}} \qquad \ldots(iii)$$

From (i), (ii) and (iii), we get

$$\therefore \quad \frac{dy}{dx} = (\sin x)^x\{x\cot x + \log\ \sin x\} + \frac{1}{2\sqrt{x(1 - x)}}$$

Q.20. If $y = \cos^{-1}\left\{\frac{3x+4\sqrt{1 - x^2}}{5}\right\}$, then find $\frac{dy}{dx}$.

Ans.

Here, $y = \cos^{-1}\left\{\frac{3x+4\sqrt{1-x^2}}{5}\right\}$

Let $x = \cos\alpha \Rightarrow \alpha = \cos^{-1} x$

$\therefore \quad y = \cos^{-1}\left\{\frac{3\cos\alpha}{5} + \frac{4}{5}\sqrt{1-\cos^2\alpha}\right\}$

$y = \cos^{-1}\left\{\frac{3}{5}\cos\alpha + \frac{4}{5}\sin\alpha\right\}$

Let $\frac{3}{5} = \cos\theta \qquad \therefore \quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

Now, $y = \cos^{-1}\{\cos\theta.\cos\alpha + \sin\theta.\sin\alpha\}$

$\therefore \quad y = \cos^{-1}(\cos(\theta-\alpha)) = \theta - \alpha$

$\Rightarrow \quad y = \cos^{-1}\frac{3}{5} - \cos^{-1}x \qquad \left[\because \quad \cos\theta = \frac{3}{5} \quad \Rightarrow \quad \theta = \cos^{-1}\frac{3}{5}\right]$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{-1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$

Q.21. If $y = \cos^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$, then find $\frac{dy}{dx}$.

Ans.

Given, $y = \cos^{-1}\left(\frac{2^x.2}{1+(2^x)^2}\right)$

Let $2^x = \tan\alpha \Rightarrow \alpha = \tan^{-1}(2^x)$

$\therefore \quad y = \cos^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right)$

$= \cos^{-1}(\sin 2\alpha) = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\alpha\right)\right) = \frac{\pi}{2} - 2\alpha$

$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1}(2^x)$

$\Rightarrow \quad \frac{dy}{dx} = 0 - 2\frac{1}{1+(2^x)^2}.\log_e 2.2^x = -\frac{2.2^x.\log_e 2}{1+4^x} = -\frac{2^{x+1}.\log_e 2}{1+4^x}$

Q.22. Find $\frac{dy}{dx}$, if $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$.

Ans.

Given, $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$

$$= \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$$

$\Rightarrow \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x} \qquad [\text{using } \sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x(1-x)}}$$

Q.23. Find $\frac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$.

Ans.

Given, $y = (\cos x)^x + (\sin x)^{1/x}$

$$y = u + v; \qquad \text{where } u = (\cos x)^x, \quad v = (\sin x)^{1/x}$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Now $u = (\cos x)^x$

Taking log both sides, we get

$$\log u = \log (\cos x)^x$$

$$\Rightarrow \quad \log u = x \log (\cos x)$$

Differentiating with respect to x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} =- \ x . \frac{1}{\cos x} \sin x + \log (\cos x)$$

$$\Rightarrow \quad \frac{du}{dx} = u\{ \log (\cos x) - \ x \tan x \}$$

$$= (\cos x)^x \{\log (\cos x) - x \tan x\}$$

Again, $v = (\sin x)^{1/x}$

Taking log both sides, we get

$$\log v = \frac{1}{x}\log(\sin x)$$

Differentiating with respect to x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{x}\cdot\frac{1}{\sin x}\cdot\cos x + \log(\sin x)\left(-\frac{1}{x^2}\right)$$

$$\frac{dv}{dx} = v\left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$$

$$= (\sin x)^{1/x}\left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$$

Putting the value of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (i), we get

$$\frac{dy}{dx} = (\cos x)^x\{\log(\cos x) - x\tan x\} + (\sin x)^{1/x}\left\{\frac{\cot x}{x} - \frac{\log\ \sin x}{x^2}\right\}$$

Q.24. Differentiate the following function with respect to $x : f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{x+2}{1-2x}\right)$.

Ans.

$$f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{x+2}{1-2x}\right) = \tan^{-1}\left(\frac{1-x}{1+x.1}\right) - \tan^{-1}\left(\frac{x+2}{1-2.x}\right)$$

$$= (\tan^{-1} 1 - \tan^{-1} x) - (\tan^{-1} x + \tan^{-1} 2) \qquad \left(\because\ \tan^{-1}\frac{a-b}{1+ab} = \tan^{-1}a - \tan^{-1}b\right)$$

$$= \tan^{-1} 1 - \tan^{-1} 2 - 2\tan^{-1} x$$

Differentiating with respect to x, we get

$$f'(x) = -\frac{2}{1+x^2}$$

Q.25. If $x^{13}y^7 = (x+y)^{20}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$.

OR

If $x^m y^n = (x+y)^{m+n}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$

Q.26. Differentiate with respect to *x* :

$$\sin^{-1}\left(\frac{2^{x+1}.3x}{1+(36)^x}\right)$$

Ans.

$$\text{Let } y = \sin^{-1}\left(\frac{2^{x+1}.3x}{1+(36)^x}\right) = \sin^{-1}\left(\frac{2.2^x.3^x}{1+(6^2)^x}\right) = \sin^{-1}\left(\frac{2.6^x}{1+(6^x)^2}\right)$$

$$\text{Let } 6^x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}(6^x)$$

$$\therefore \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \quad \Rightarrow \quad y = \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow \quad y = 2\theta \quad \Rightarrow \quad y = 2 . \tan^{-1}(6^x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+(6^x)^2}.\log_e 6.6^x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2.6^x.\log_e 6}{1+36^x}$$

Q.27. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ **with respect to $\tan^{-1} x$, when $x \neq 0$.**

Ans.

$$\text{Let } u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) \text{ and } v = \tan^{-1} x$$

We have to find $\frac{du}{dv}$

$$\text{Now, } u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

$$\text{Let } x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1} x$$

$$\therefore \quad u = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right] = \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right] = \tan^{-1}\left[\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\cos\theta}.\frac{\cos\theta}{\sin\theta}\right] = \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right] = \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right] = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$\therefore \quad u = \frac{1}{2}\tan^{-1} x$

Differentiating both sides with respect to x, we get

$$\frac{du}{dx} = \frac{1}{2(1+x^2)} \quad \ldots(i)$$

Also, $v = \tan^{-1} x$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{1+x^2} \quad \ldots(ii)$$

$$\therefore \quad \frac{du}{dv} = \frac{1}{2(1+x^2)} \times \frac{1+x^2}{1} = \frac{1}{2}$$

Q.28. If $x \sin(a + y) + \sin a \cos(a + y) = 0$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Ans.

Given $x \sin(a + y) + \sin a \cos(a + y) = 0$

$$\Rightarrow \quad x = -\frac{\sin a . \cos(a+y)}{\sin(a+y)} \quad \Rightarrow \quad x = -\sin a . \cot(a + y)$$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = +\sin a . \operatorname{cosec}^2(a+y) = \frac{\sin a}{\sin^2(a+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Q.29. If $e^x + e^y = e^{x+y}$, then prove that $\frac{dy}{dx} + e^{y-x} = 0$.

Ans.

Given, $e^x + e^y = e^{x+y}$

Differentiating both sides with respect to x, we get

$$e^x + e^y.\frac{dy}{dx} = e^{x+y}\left\{1+\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad e^x + e^y.\frac{dy}{dx} = e^{x+y} + e^{x+y}.\frac{dy}{dx} \qquad \Rightarrow \qquad (e^{x+y} - e^y)\frac{dy}{dx} = e^x - e^{x+y}$$

$$\Rightarrow \quad (e^x + e^y - e^y)\frac{dy}{dx} = e^x - e^x - e^y \qquad [\because\ e^x + e^y = e^{x+y}(\text{ given })]$$

$$\Rightarrow \quad e^x.\frac{dy}{dx} = -e^y \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{e^y}{e^x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -e^{y-x} \qquad \Rightarrow \qquad \frac{dy}{dx} + e^{y-x} = 0$$

Q.30. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.

Ans.

We have

$$x = e^{\cos 2t}$$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = e^{\cos 2t}(-\ 2\sin 2t) = 2x\sin 2t$$

Again $y = e^{\sin 2t}$

Differentiating w.r.t. t, we get

$$\frac{dy}{dt} = e^{\sin 2t}.2\cos 2t = 2y\cos 2t$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2y\cos 2t}{-2x\sin 2t}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-y\ \cos\ 2t}{x\ \sin\ 2t} \qquad [\because x = e^{\cos 2t} \Rightarrow \log x = \cos 2t;\ y = e^{\sin 2t} \Rightarrow \log y = \sin 2t]$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{y\ \log\ x}{x\ \log y}$

Hence proved.

Long Answer Questions-I-B (OIQ)

[4 Mark]

Q.1. If $y = \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$, then find $\frac{dy}{dx}$.

Ans.

Given, $y=\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$

Taking log on both sides, we get

$$\log y=\frac{1}{2}[\log(x-3)+\log(x^2+4)-\log(3x^2+4x+5)]$$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x-3}+\frac{1}{x^2+4}\times 2x-\frac{1}{3x^2+4x+5}\times(6x+4)\right]$$

$$\therefore\quad \frac{dy}{dx}=\frac{y}{2}\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

$$\frac{dy}{dx}=\frac{1}{2}\left(\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\right)\left[\frac{1}{x-3}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

Q.2. Find the derivative of y with respect to x, where $y = (x)^{\sin x} + (\sin x)^x$.

Ans.

Given, $y=(x)^{\sin x}+(\sin x)^x$ $\quad\ldots(i)$

Let $u=x^{\sin x}$, and $v=(\sin x)^x$, then (i) becomes $y=u+v$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\quad\ldots(ii)$$

First consider, $u=x^{\sin x}$

Taking log on both sides, we get $\log u=\sin x\,.\log x$

Differentiating with respect to x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\cos x.\log x+\sin x.\frac{1}{x}\quad\Rightarrow\quad\frac{du}{dx}=u\left(\cos x\log x+\frac{\sin x}{x}\right)$$

$$\Rightarrow\quad\frac{du}{dx}=(x)^{\sin x}\left[\cos x(\log x)+\frac{\sin x}{x}\right]\quad\ldots(iii)$$

Again consider, $v = (\sin x)^x$

Taking log on both sides, we get $\log v = x \log \sin x$

Differentiating with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = 1.\log\ \sin x + x.\frac{1}{\sin x}.\cos x \quad \Rightarrow \quad \frac{dv}{dx} = v(\log\ \sin x + x\cot x)$$

$$\frac{dv}{dx} = (\sin x)^x[\log\ \sin x + x\cot x] \qquad \ldots(iv)$$

From (ii), (iii) and (iv), we get

$$\frac{dy}{dx} = (x)^{\sin x}\left[\cos x(\log x) + \frac{\sin x}{x}\right] + (\sin x)^x[\log\ \sin x + x\cot x]$$

Q.3. **If $y = [x + \sqrt{x^2 + a^2}]^n$, then prove that $\frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$.**

We have, $y = [x + \sqrt{x^2 + a^2}]^n \quad \Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}[\{x + \sqrt{x^2 + a^2}\}^n]$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}\frac{d}{dx}\{x + \sqrt{x^2 + a^2}\} \qquad \text{[By chain rule]}$$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}.\left\{\frac{d}{dx}(x) + \frac{d}{dx}\sqrt{x^2 + a^2}\right\}$$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}.\left\{1 + \frac{1}{2}(x^2 + a^2)^{\frac{-1}{2}}.\frac{d}{dx}(x^2 + a^2)\right\}$$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}.\left\{1 + \frac{1}{2\sqrt{x^2+a^2}}.2x\right\} = n\{x + \sqrt{x^2 + a^2}\}^{n-1}.\left\{1 + \frac{x}{\sqrt{x^2+a^2}}\right\}$$

$$= n\{x + \sqrt{x^2 + a^2}\}^{n-1}\left\{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right\} = \frac{n\{x+\sqrt{x^2+a^2}\}^n}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$$

Q.4.

If $y\sqrt{x^2 + 1} = \log(\sqrt{x^2 + 1} - x)$, then show that $(x^2 + 1)\frac{dy}{dx} + xy + 1 = 0$.

Ans.

Given $y\sqrt{x^2+1} = \log\left(\sqrt{x^2+1} - x\right)$

Differentiating with respect to x on both sides, we get

$$y \times \frac{1}{2\sqrt{x^2+1}} \times 2x + \sqrt{x^2+1}\frac{dy}{dx} = \frac{1}{(\sqrt{x^2+1} - x)} \times \left(\frac{1}{2\sqrt{x^2+1}} \times 2x - 1\right)$$

$$\Rightarrow \quad \frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}\frac{dy}{dx} = \frac{\left(\frac{x}{\sqrt{x^2+1}} - 1\right)}{(\sqrt{x^2+1} - x)}$$

$$\Rightarrow \quad \frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}.\frac{dy}{dx} = \frac{(x - \sqrt{x^2+1})}{(\sqrt{x^2+1})(\sqrt{x^2+1} - x)}$$

$$\Rightarrow \quad \frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}.\frac{dy}{dx} = \frac{-1}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad \sqrt{x^2+1}\frac{dy}{dx} = \frac{-1}{\sqrt{x^2+1}} - \frac{xy}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad \sqrt{x^2+1}\frac{dy}{dx} = \frac{-(1+xy)}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad (x^2+1)\frac{dy}{dx} = -(1+xy)$$

Hence, $(x^2+1)\frac{dy}{dx} + xy + 1 = 0$

Q.5. Find $\frac{dy}{dx}$: $y = (\sin x)^x + (\cos x)^{\tan x}$.

Ans.

Given, $y = (\sin x)^x + (\cos x)^{\tan x}$

Let $u = (\sin x)^x$ and $v = (\cos x)^{\tan x}$

We have, $y = u + v$ then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(i)$$

Now, $u = (\sin x)^x$

Taking log on both sides, we get $\log u = x \log \sin x$

Differentiating both sides, with respect to x, we have

$$\frac{1}{u}\frac{du}{dx} = x \times \frac{1}{\sin x} \times \cos x + \log \sin x$$

$$\therefore \quad \frac{du}{dx} = u(x \cot x + \log \sin x)$$

$$\frac{du}{dx} = (\sin x)^x (x \cot x + \log \sin x) \quad ...(ii)$$

Again, $v = (\cos x)^{\tan x}$

Taking log on both sides, we get

$$\log v = \tan x [\log \cos x]$$

Differentiating both sides with respect to x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx}=\tan x\times\frac{1}{\cos x}\times(-\sin x)+\log\cos x\times\sec^2 x$$

$$=-\tan^2 x+\sec^2 x\log\cos x$$

$$\frac{dv}{dx}=v(\sec^2 x\log\cos x-\tan^2 x)$$

$$=(\cos x)^{\tan x}(\sec^2 x\log\cos x-\tan^2 x) \qquad ...(iii)$$

From (i), (ii) and (iii), we have

$$\frac{dy}{dx}=(\sin x)^x(x\cot x+\log\sin x)+(\cos x)^{\tan x}(\sec^2 x\log\cos x-\tan^2 x)$$

Q.6.

If $x\in R-[-1,1]$ then prove that the derivative of $\sec^{-1}x$ with respect to x is $\frac{1}{|x|\sqrt{x^2-1}}$.

Ans.

Let $y=\sec^{-1}x$

Then, $\sec y=\sec(\sec^{-1}x)=x$

Differentiating both sides with respect to x, we have

$$\Rightarrow \frac{d}{dx}\sec y=\frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(\sec y)\frac{dy}{dx}=1$$

$$\Rightarrow \sec y\tan y\frac{dy}{dx}=1 \qquad \text{[Using chain rule]}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sec y\tan y}=\frac{1}{|\sec y|.|\tan y|}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{|\sec y|\sqrt{\tan^2 y}}=\frac{1}{|\sec y|\sqrt{\sec^2 y-1}}=\frac{1}{|x|\sqrt{x^2-1}}$$

$$\left[\begin{array}{l}\text{If } x1, \text{ then } y \in \left(0, \frac{\pi}{2}\right) \\ \therefore \quad \sec y > 0, \ \tan y > 0 \\ \Rightarrow \quad |\sec y| . |\tan y| = \sec y \tan y \\ \text{If } x - 1, \text{ then} \\ y \in \left(\frac{\pi}{2}, \pi\right) \therefore \quad \sec y < 0, \ \tan \ y < 0 \\ \Rightarrow \quad |\sec y| \tan y| \\ \Rightarrow \quad (-\sec y)(-\tan y) = \sec y \tan y\end{array}\right]$$

Long Answer Questions-I-C (PYQ)

[4 Marks]

Q.1. If *x* = *a* cos θ + *b* sin θ and *y* = *a* sin θ – *b* cos θ, then show that

$y^2\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0.$

Ans.

Given, $x = a\cos\theta + b\sin\theta$

$\Rightarrow \quad \frac{dx}{d\theta} = -a\sin\theta + b\cos\theta \qquad \ldots(i)$

Also, $y = a\sin\theta - b\cos\theta$

$\frac{dy}{d\theta} = a\cos\theta + b\sin\theta \qquad \ldots(ii)$

$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta + b\sin\theta}{-a\sin\theta + b\cos\theta} \qquad$ [From (i) and (ii)]

$\Rightarrow \quad \frac{dy}{dx} = \frac{a\cos\theta + b\sin\theta}{b\cos\theta - a\sin\theta} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \qquad \ldots(iii)$

Differentiating again with respect to x, we get

$\frac{d^2y}{dx^2} = -\frac{y - x.\frac{dy}{dx}}{y^2}$

$\Rightarrow \quad y^2\frac{d^2y}{dx^2} = -y + x\frac{dy}{dx}$

$\Rightarrow \quad y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Q.2. If $y = Pe^{ax} + Qe^{bx}$, then show that $\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$

Ans.

Given, $y = Pe^{ax} + Qe^{bx}$

On differentiating with respect to x, we have

$$\frac{dy}{dx} = Pae^{ax} + Qbe^{bx}$$

Again, differentiating with respect to x, we have

$$\frac{d^2y}{dx^2} = Pa^2 e^{ax} + Qb^2 e^{bx}$$

Now, LHS $= \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby$

$= Pa^2 e^{ax} + Qb^2 e^{bx} - (a+b)\left(Pae^{ax} + Qbe^{bx}\right) + ab\left(Pe^{ax} + Qe^{bx}\right)$

$= Pa^2 e^{ax} + Qb^2 e^{bx} - Pa^2 e^{ax} - Pabe^{ax} - Qabe^{bx} - Qb^2 e^{bx} + Pabe^{ax} + Qabe^{bx}$

$= 0 =$ RHS

Q.3. **If $y = \sin(\log x)$, then prove that $x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$.**

Ans.

Given, $y = \sin(\log x)$

$\Rightarrow \quad \frac{dy}{dx} = \cos(\log x) \times \frac{1}{x} = \frac{\cos(\log x)}{x}$

Again, $\frac{d^2y}{dx^2} = \frac{x\left[-\sin(\log x)\times\frac{1}{x}\right] - \cos(\log x)}{x^2} = \frac{-\cos(\log x) - \sin(\log x)}{x^2}$

Now, LHS $= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y$

$= \frac{x^2\{-\cos(\log x) - \sin(\log x)\}}{x^2} + \frac{x\cos(\log x)}{x} + \sin(\log x)$

$= -\cos(\log x) - \sin(\log x) + \cos(\log x) + \sin(\log x) = 0 =$ RHS

Q.4. **If $y = \log[x + \sqrt{x^2+1}]$, then prove that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$**

Ans.

Given, $y = \log [x + \sqrt{x^2 + 1}]$

Differentating with respect to x, we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}} \times \left[1 + \frac{2x}{2\sqrt{x^2+1}}\right] = \frac{(x+\sqrt{x^2+1})}{(x+\sqrt{x^2+1})\times\sqrt{x^2+1}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = -\ \frac{1}{2}(x^2+1)^{-\ 3/2}.2x = \frac{-\ x}{(x^2+1)^{3/2}}$$

$$\Rightarrow \quad (x^2+1)\frac{d^2y}{dx^2} = \frac{-\ x}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad (x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Q.5. If $y = \frac{\sin^{-1} x}{\sqrt{1\ -\ x^2}}$, then show that $(1 -\ x^2)\frac{d^2y}{dx^2}\ -\ 3x\frac{dy}{dx}\ -\ y = 0.$

Ans.

Given, $y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{\sqrt{1 - x^2}.\frac{1}{\sqrt{1 - x^2}} - \sin^{-1} x.\frac{-2x}{2\sqrt{1 - x^2}}}{(\sqrt{1 - x^2})^2} = \frac{1 + xy}{1 - x^2} \qquad \ldots(i)$$

Again differentiating with respect to x, we get

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{(1 - x^2).\left(x.\frac{dy}{dx} + y\right) + (1 + xy).2x}{(1 - x^2)^2}$$

$$\Rightarrow \quad (1 - x^2).\frac{d^2y}{dx^2} = x.\frac{dy}{dx} + y + \frac{(1 + xy).2x}{1 - x^2}$$

$$\Rightarrow \quad (1 - x^2).\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y + 2x\frac{dy}{dx} \qquad \text{[using } (i)\text{]}$$

$$\Rightarrow \quad (1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$$

Q.6. **If $y = e^x (\sin x + \cos x)$, then show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$**

Ans.

Given, $y = e^x (\sin x + \cos x)$

$$\therefore \quad \frac{dy}{dx} = e^x (\cos x - \sin x) + (\sin x + \cos x).e^x$$

$$\Rightarrow \quad \frac{dy}{dx} = 2e^x \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = 2e^x \cos x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2(-e^x \sin x + \cos x.e^x$$

$$= -2e^x \sin x + 2e^x \cos x = -2e^x \sin x - 2e^x \cos x + 4e^x \cos x$$

Now, $= -2e^x(\sin x + \cos x) + 2.(2e^x \cos x) = -2y + 2\frac{dy}{dx}$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Q.7. If $y = \operatorname{cosec}^{-1} x$, $x > 1$, then show that $x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$

Ans.

$\because \quad y = \operatorname{cosec}^{-1} x$

Differentiating with respect to x, we get

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{x\sqrt{x^2-1}\,.\,0 + 1.\left\{x\,.\,\frac{2x}{2\sqrt{x^2-1}} + \sqrt{x^2-1}\right\}}{x^2(x^2-1)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{x^2+x^2-1}{x^2(x^2-1)\,.\,\sqrt{x^2-1}} = \frac{2x^2-1}{\sqrt{x^2-1}\,.\,x^2(x^2-1)}$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2} = \frac{2x^2-1}{x\sqrt{x^2-1}} = (2x^2-1)\left(-\frac{dy}{dx}\right)$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$$

Q.8. If $y = \sin^{-1} x$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$

Ans.

$\because \quad y = \sin^{-1} x$

Differentiating with respect to x, we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = 1$$

Again differentiating with respect to x, we get

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{1\times(-2x)}{2\sqrt{1-x^2}} = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - \frac{xdy}{dx} = 0$$

Q.9. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

Ans.

Given, $y = 3\cos(\log x) + 4\sin(\log x)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$$\Rightarrow \quad y_1 = \frac{1}{x}[-3\sin(\log x) + 4\cos(\log x)]$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{x\left[\frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}\right] - [-3\sin(\log x) + 4\cos(\log x)]}{x^2}$$

$$= \frac{-3\cos(\log x) - 4\sin(\log x) + 3\sin(\log x) + 4\cos(\log x)}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

$\Rightarrow \quad y_2 = \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$

Now, LHS $= x^2 y_2 + x y_1 + y$

$= x^2 \left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x \times \frac{1}{x}[-3\sin(\log x) + 4\cos(\log x)] + 3\cos(\log x) + 4\sin(\log x)$

$= -\sin(\log x) - 7\cos(\log x) - 3\sin(\log x) + 4\cos(\log x) + 3\cos(\log x) + 4\sin(\log x)$

$= 0 =$ RHS

Q.10.

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, $0 < t < \frac{\pi}{2}$, find $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

Ans.

Given, $x = a(\cos t + t\sin t)$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$$

$$\Rightarrow \quad \frac{dx}{dt} = at\cos t \qquad \ldots(i)$$

Differentiating again with respect to t, we get

$$\frac{d^2x}{dt^2} = a(-t\sin t + \cos t) = a(\cos t - t\sin t)$$

Again, $y = a(\sin t - t\cos t)$

Differentiating with respect to t, we get

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) \qquad \ldots(ii)$$

$$\Rightarrow \quad \frac{dy}{dt} = at \ \sin \ t$$

Differentiating again with respect to t we get

$$\frac{d^2y}{dt^2} = a(t\cos t + \sin t)$$

Now, $\frac{dy}{dx} = \frac{dt/dy}{dt/dx}$ [from (i) and (ii)]

$$\Rightarrow \quad \frac{dy}{dx} = \frac{at \ \sin \ t}{at \ \cos \ t}$$

$$\Rightarrow \quad \frac{dy}{dx} = \tan t$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t.\frac{dt}{dx} = \sec^2 \ t.\frac{1}{dt/dx} = \frac{\sec^2 t}{at \ \cos \ t} \qquad \text{[from } (i)\text{]}$$

$$= \frac{\sec^3 t}{at}$$

Hence $\frac{d^2x}{dt^2} = a(\cos t - t\sin t)$

and $\frac{d^2y}{dt^2} = a\,(t\cos t + \sin t)$ and $\frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}$

Q.11. If $x = a\left(\cos t + \log \ \tan \frac{t}{2}\right), y = a \ \sin t,$ then find $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

Ans.

Given, $x = a\left(\cos t + \log\ \tan\frac{t}{2}\right)$

Differentiating with respect to *t*, we get

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right)$$

$$= a\left\{-\sin t + \frac{1}{2\sin\frac{t}{2}.\cos\frac{t}{2}}\right\} = a\left\{-\sin t + \frac{1}{\sin t}\right\}$$

$$\frac{dx}{dt} = a\left(\frac{1-\sin^2 t}{\sin t}\right) = a\frac{\cos^2 t}{\sin t}$$

$\because \quad y = a\sin t$

Differentiating with respect to *t*, we get

$$\frac{dy}{dt} = a.\cos t \quad \Rightarrow \quad \frac{d^2y}{dt^2} = -a\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t.\sin t}{a\cos^2 t} = \tan t$$

$$\therefore \quad \frac{d^2y}{dx^2} = \sec^2 t.\frac{dt}{dx} = \sec^2 t.\frac{1\times\sin t}{a\cos^2 t} = \frac{1}{a}\sec^4 t.\sin t$$

Hence, $\frac{d^2y}{dx^2} = -a\sin t$ and $\frac{d^2y}{dx^2} = \frac{\sec^4 t\sin t}{a}$

Q.12. If $x = a\sin t$ and $y = a\left(\cos t + \log\ \tan\frac{t}{2}\right)$, then find $\frac{d^2y}{dx^2}$.

Ans.

Given, $x = a \sin t$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = a \cos t \quad ...(i)$$

Again, $\because y = a\left[\cos t + \log\left(\tan \frac{t}{2}\right)\right]$

Differentiating both sides with respect to t, we get

$$\frac{dy}{dt} = a\left[-\sin t + \frac{1}{\tan \frac{t}{2}}.\sec^2 \frac{t}{2}.\frac{1}{2}\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{a\,(1-\sin^2 t)}{\sin t}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{a \cos^2 t}{\sin t} \quad ...(ii)$$

$$\because \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a \cos^2 t}{\sin t} \times \frac{1}{a \cos t} \quad \text{[From } (i) \text{ and } (ii)\text{]}$$

$$\frac{dy}{dx} = \cot t$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 t.\frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\operatorname{cosec}^2 t.\frac{1}{a \cos t} = \frac{-\operatorname{cosec}^2 t}{a \cos t}$$

Q.13. If $= \log\left[x + \sqrt{x^2 + a^2}\right]$, show that $(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$

Ans.

Given $y = \log\,[x + \sqrt{x^2 + a^2}]$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}} \cdot \left[1 + \frac{2x}{2\sqrt{x^2+a^2}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+\sqrt{x^2+a^2}}{(x+\sqrt{x^2+a^2})(\sqrt{x^2+a^2})}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2+a^2}} \qquad \ldots(i)$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2 + a^2)^{-\frac{3}{2}}.2x = \frac{-x}{(x^2+a^2)^{\frac{3}{2}}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-x}{(x^2+a^2).\sqrt{x^2+a^2}}$$

$$\Rightarrow \quad (x^2 + a^2)\frac{d^2y}{dx^2} = -\frac{x}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \quad (x^2 + a^2)\frac{d^2y}{dx^2} + x.\frac{dy}{dx} = 0 \qquad [\text{from } (i)]$$

Q.14. **If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$.**

Ans.

Given, $x = a\cos^3\theta$

Differentiating both sides with respect to θ, we get

$$\frac{dx}{d\theta} = -3a\cos^2\theta.\sin\theta \qquad ...(i)$$

Also, $y = a\sin^3\theta$

Differentiating both sides with respect to θ, we get

$$\frac{dy}{d\theta} = 3a\sin^2\theta.\cos\theta \qquad ...(ii)$$

Now $\frac{dy}{dx} = \frac{d\theta/dy}{d\theta/dx} = \frac{3a\ \sin^2\theta.\cos\theta}{-3a\ \cos^2\theta.\sin\theta}$

$\Rightarrow \quad \frac{dy}{dx} = -\tan\theta$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\sec^2\theta.\frac{d\theta}{dx}$

$$= \frac{-\sec^2\theta}{-3a\ \cos^2\theta.\sin\theta}$$

$$= \frac{1}{3a}\sec^4\theta.\ \text{cosec}\ \theta$$

$\therefore \quad \left.\frac{d^2y}{dx^2}\right]_{x=\frac{\pi}{6}} = \frac{1}{3a}\sec^4\frac{\pi}{6}.\ \text{cosec}\ \frac{\pi}{6}$

$$= \frac{1}{3a}.\left(\frac{2}{\sqrt{3}}\right)^4 \times 2 = \frac{32}{27a}$$

Q.15. **If $y = x^x$, then prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$**

Ans.

Given, $y = x^x$

Taking logarithm on both sides, we get

$$\log y = x . \log x$$

Differentiating both sides, we get

$$\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = x . \frac{1}{x} + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1 + \log x) \quad \ldots(i)$$

Again differentiating both sides, we get

$$\frac{d^2y}{dx^2} = y . \frac{1}{x} + (1 + \log x) . \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y} \cdot \frac{dy}{dx} \cdot \frac{dy}{dx} \quad [\text{From } (i)]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$

Q.16. If $y = x^3 \log\left(\frac{1}{x}\right)$, then prove that $x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3x^2 = 0.$

Ans.

The given differential equation is $y = x^3 \log\left(\frac{1}{x}\right)$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = x^3.x\left(\frac{-1}{x^2}\right) + \log\ \frac{1}{x}.3x^2 = -\ x^2 + 3x^2.log\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = -\ x^2 + \frac{3}{x}.x^3 \log\ \left(\frac{1}{x}\right)$$

$$\Rightarrow \quad x\frac{\mathrm{dy}}{\mathrm{dx}} = -\ x^3 + 3y$$

Again differentiating with respect to x

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} + x^2\frac{d^2y}{\mathrm{dx}^2} = -3x^2 + 3\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\Rightarrow \quad x\frac{d^2y}{\mathrm{dx}^2} - 2\frac{\mathrm{dy}}{\mathrm{dx}} + 3x^2 = 0$$

Hence proved.

Q.17. **If $\mathrm{y} = \left(\mathrm{x} + \sqrt{1 + \mathrm{x}^2}\right)^{\mathrm{n}}$, then show that**

$$(1 + x^2)\frac{d^2y}{\mathrm{dx}^2} + x\frac{\mathrm{dy}}{\mathrm{dx}} = n^2y$$

Ans.

Given $y = \left(x + \sqrt{1 + x^2}\right)^n$

Differentiating with respect to *x*, we get

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = n\,(x + \sqrt{1 + x^2})^{n-1}.\left\{1 + \frac{2x}{2\sqrt{1+x^2}}\right\}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} \quad = n\,(x + \sqrt{1 + x^2})^{n-1}.\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{ny}}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \sqrt{1 + x^2}.\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{ny}$$

Again differentiating with respect to *x*, we get

$$\sqrt{1 + x^2}.\frac{d^2y}{\mathrm{dx}^2} + \frac{2x}{2\sqrt{1+x^2}}.\frac{\mathrm{dy}}{\mathrm{dx}} = n\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\Rightarrow \quad (1 + x^2)\frac{\mathrm{dy}}{\mathrm{dx}} + x.\frac{\mathrm{dy}}{\mathrm{dx}} = n\ .\sqrt{1 + x^2}.\frac{\mathrm{dy}}{\mathrm{dx}}$$

$$\Rightarrow \quad (1 + x^2)\frac{\mathrm{dy}}{\mathrm{dx}} + x\frac{\mathrm{dy}}{\mathrm{dx}} = n.\sqrt{1 + x^2}.\frac{\mathrm{ny}}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad (1 + x^2)\frac{d^2y}{\mathrm{dx}^2} + x\frac{\mathrm{dy}}{\mathrm{dx}} = n^2y$$

Long Answer Questions-I-D (PYQ)

[4 Marks]

Q.1. Verify Lagrange's mean value theorem for the following function:

$f(x) = x^2 + 2x + 3$, for [4, 6].

Ans.

$f(x) = x^2 + 2x + 3$ for $[4, 6]$

i. Given function is a polynomial hence it is continuous.

ii. $f'(x) = 2x + 2$ which is differentiable.

$f(4) = 16 + 8 + 3 = 27$ and $f(6) = 36 + 12 + 3 = 51$

$\Rightarrow$ $f(4) \neq f(6)$. All conditions of mean value theorem are satisfied.

$\therefore$ There exist at least one real value $c \in (4, 6)$

such that $f'(c) = \frac{f(6)-f(4)}{6-4} = \frac{24}{2} = 12$

$\Rightarrow$ $2c + 2 = 12$ or $c = 5 \in (4, 6)$

Hence, Lagrange' mean value theorem is verified.

Q.2. Verify Mean Value theorem for the function $f(x) = 2 \sin x + \sin 2x$ on [0, π].

Ans.

We have,

$f(x) = 2 \sin x + \sin 2x$

$f(x)$ is continuous in [0, π] being trigonometric function.

Also $f(x)$ is differentiable on (0, π).

Hence, condition of Mean Value theorem is satisfied.

Therefore, mean value theorem is applicable.

So, ∃ a real number c such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} \qquad \ldots(i)$$

Now $f(0) = 2 \sin 0 + \sin 0 = 0$

$f(p) = 2 \sin \pi + \sin 2\pi = 0$

and $f'(x) = 2 \cos x + 2 \cos 2x$

∴ $f'(c) = 2 \cos c + 2 \cos 2c$

From (i)

$$2 \cos c + 2 \cos 2c = \frac{0 - 0}{\pi}$$

⇒ $2 \cos c + 2 \cos 2c = 0$

⇒ $2 \cos c + 2(2 \cos^2 c - 1) = 0$

⇒ $\cos c + 2 \cos^2 c - 1 = 0$

⇒ $2 \cos^2 c + \cos c - 1 = 0$

⇒ $2 \cos^2 c + 2 \cos c - \cos c - 1 = 0$

⇒ $2 \cos c (\cos c + 1) - 1 (\cos c + 1) = 0$

$\Rightarrow \quad (\cos c + 1)(2\cos c - 1) = 0$

$\Rightarrow \quad \cos c = -1$ and $\cos c = \frac{1}{2}$

$\Rightarrow \quad c = \pi$ and $c = \frac{\pi}{3}$

$\because \quad c = \frac{\pi}{3} \in (0, \pi)$

Hence Mean Value theorem is verified.

Long Answer Questions-I-D (OIQ)

[4 Marks]

Q.1. **Verify Lagrange's mean value theorem for the function** $f(x) = x + \frac{1}{2}$ in $[1, 3]$.

Ans.

Given, $f(x) = x + \frac{1}{x}$ or $f(x) = \frac{x^2+1}{x}$

i. Since $f(x)$ is a rational function such that the denominator is not zero for any value in $[1, 3]$, it is a continuous function.

ii. $f'(x) = 1 - \frac{1}{x^2}$ which exist in $(1, 3)$ $\quad \therefore \quad f(x)$ is differentiable in $(1, 3)$

Thus, all the conditions of Lagrange's Mean Value theorem are satisfied. Hence, there exist at least one real value c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad ...(i)$$

where $f'(c) = 1 - \frac{1}{c^2}; f(b) = f(3) = \frac{10}{3}$ and $f(a) = f(1) = 2 \qquad ...(ii)$

From (i) and (ii), we get

$$\Rightarrow \quad 1-\frac{1}{c^2}=\frac{\frac{10}{3}-2}{3-1}$$

$$\Rightarrow \quad \frac{c^2-1}{c^2}=\frac{2}{3}$$

$$\Rightarrow \quad 3c^2-3=2c^2$$

$$\Rightarrow \quad c^2=3$$

$$\Rightarrow \quad c=\pm\sqrt{3}$$

Neglecting $c=-\sqrt{3}$ as $-\sqrt{3}\notin(1,3)$ $\therefore$ $c=\sqrt{3}\in(1,3)$

Hence, Lagrange's mean value theorem is verified.

Q.2. Using Rolle's theorem, find the points on the curve $y = x^2$, where $x \in [-2, 2]$ and the tangent is parallel to x-axis.

Ans.

$f(x) = x^2$

i. $f(x)$ is a polynomial, hence continuous in $[-2, 2]$

ii. $f'(x) = 2x$ which exist in $[-2, 2]$

$\therefore$ $f(x)$ is differentiable in $[-2, 2]$

iii. $f(-2) = (-2)^2 = 4$

$f(2) = (2)^2 = 4$

$\therefore \quad f(2) = f(-2)$

Thus, all the conditions of Rolle's theorem are applicable, then there exist at least one real value c, such that

$f'(c) = 0$

$\Rightarrow \quad 2c = 0$

$\Rightarrow \quad c = 0$

when $x = 0, y = (0)^2 = 0$

$\therefore \quad (0, 0)$ is the required point.