Very Short Answer Questions (PYQ)

[1 Mark]

Q.1. If the rate of change of volume of a sphere is equal to the rate of change of its radius, find the radius of the sphere.

Ans.

We have, $\frac{dV}{dt} = \frac{dr}{dt}$, where V is the volume and r is the radius of the sphere

$$\Rightarrow \frac{d\frac{4}{3}\pi r^3}{dt} = \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{4}{3}\pi \times 3r^2 \frac{dr}{dt} = \frac{dr}{dt}$$

$$\Rightarrow 4\pi r^2 = 1$$

$$\Rightarrow \quad r^2 = \frac{1}{4\pi}$$

$$\therefore r = \frac{1}{2\sqrt{\pi}} \text{units}$$

Q.2. An edge of a variable cube is increasing at the rate of 5 cm per second. How fast is the volume increasing when the side is 15 cm?

Ans.

Let x be the edge of the cube and V be the volume of the cube at any time t.

Given, $\frac{dx}{dt} = 5\,\text{cm/s}, x = 15\,\text{cm}$

Since we know the volume of cube = $(\text{side})^3$ *i.e.,* $V = x^3$.

$$\Rightarrow \quad \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = 3 \cdot (15)^2 \times 5 = 3375 \text{ cm}^3/\text{sec}$$

Q.3. Find the slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at $t = 2$.

Ans.

$$\text{Slope of the tangent} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t-2}{2t+3}$$

$$\Rightarrow \frac{dy}{dx} \text{ at } t=2 = \left(\frac{4t-2}{2t+3}\right)_{\text{at } t=2} = \frac{4(2)-2}{2(2)+3} = \frac{6}{7}$$

Q.4. If $y = \log_e x$, then find Δy when $x = 3$ and $\Delta x = 0.03$.

Ans.

We have, $y = \log_e x$

$$\Delta y = \frac{\mathrm{dy}}{\mathrm{dx}} \cdot \Delta x = \frac{1}{x} \cdot \Delta x = \frac{0.03}{3} = 0.01$$

Q.5. Find the rate of change of the area of a circle with respect to its radius '*r*' when *r* = 4 cm.

Ans.

If A is area and *r* is the radius of a circle, then

$$A = \pi r^2 \qquad \Rightarrow \qquad \frac{\mathrm{dA}}{\mathrm{dr}} = 2\pi r$$

$$\therefore \qquad \left[\frac{\mathrm{dA}}{\mathrm{dr}}\right]_{r=4} = 8\pi \text{ cm}^2 / \text{cm}$$

Short Answer Questions (PYQ)

[2 Mark]

Q.1. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue when $x = 5$.

Ans.

Given: $R(x) = 3x^2 + 36x + 5$

$\Rightarrow \ R'(x) = 6x + 36$

$\therefore$ Marginal revenue $(\text{when } x = 5) = R'(x)]_{x} = 5$

$= 6 \times 5 + 36 = ₹\ 66.$

Q.2. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added.

Ans.

We have to find $[P(x)]x = 3$

Now, $P(x) = 0.005x^3 + 0.02x^2 + 30x$

$\therefore P(x) = 0.015x^2 + 0.04x + 30$

$\Rightarrow [P(x)]x = 3 = 0.015 \times 9 + 0.04 \times 3 + 30$

$= 0.135 + 0.12 + 30 = 30.255$

Q.3. If $C = 0.003x^3 + 0.02x^2 + 6x + 250$ gives the amount of carbon pollution in air in an area on the entry of x number of vehicles, then find the marginal carbon pollution in the air, when 3 vehicles have entered in the area.

Ans.

We have to find $[C(x)]x = 3$

Now $C(x) = 0.003x^3 + 0.02x^2 + 6x + 250$

$\therefore C(x) = 0.009x^2 + 0.04x + 6$

$[C(x)]x = 3 = 0.009 \times 9 + 0.04 \times 3 + 6 = 0.081 + 0.12 + 6 = 6.201$

Q.4. The contentment obtained after eating *x*-units of a new dish at a trial function is given by the Function $C(x) = x^3 + 6x^2 + 5x + 3$. If the marginal contentment is defined as rate of change of $C(x)$ with respect to the number of units consumed at an instant, then find the marginal contentment when three units of dish are consumed.

Ans.

Given, $C(x) = x^3 + 6x^2 + 5x + 3$

$\Rightarrow \; C(x) = 3x^2 + 12x + 5$

$\Rightarrow \; [C(x)]x = 3 = 3 \times 9 + 36 + 5 = 27 + 36 + 5 = 68$ units

$\therefore$ Required marginal contentment $[C(x)]x = 3 = 68$ units

Q.5. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, find the approximate error in calculating its surface area.

Ans.

Here, radius of the sphere $r = 9$ cm.

Error in calculating radius, $\delta r = 0.03$ cm.

Let δs be approximate error in calculating surface area.

If S be the surface area of sphere, then $S = 4\pi r^2$

$\Rightarrow \; \frac{dS}{dr} = 4\pi.\, 2r = 8\pi r$

Now by definition, approximately

$$\frac{dS}{dr} = \frac{\delta S}{\delta r}$$

$$\left[\because \frac{dS}{dr} = \lim_{\delta r \to 0} \frac{\delta s}{\delta r}\right]$$

$$\Rightarrow \delta s = \left(\frac{dS}{dr}\right).\delta r \quad \Rightarrow \quad \delta s = 8\pi r\,.\,\delta r$$

$= 8\pi \times 9 \times 0.03 \text{ cm}^2 = 2.16\text{p cm}^2$

$[\because r = 9 \text{ cm}]$

Q.6. Using differentials, find the approximate value of √49.5.

Ans.

Let $f(x) = \sqrt{x,}$ where $x = 49$ and $\delta x = 0.5$

$$\therefore\ f(x + \delta x) = \sqrt{x + \delta x} = \sqrt{49.5}$$

Now by definition, approximately we can write

$$f'(x) = \frac{f(x+\delta x) - f(x)}{\delta x} \qquad \dots(\text{i})$$

Here $f(x) = \sqrt{x} = \sqrt{49} = 7$ and $\delta x = 0.5$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{49}} = \frac{1}{14}$$

Putting these values in (i), we get

$$\frac{1}{14} = \frac{\sqrt{49.5}-7}{0.5}$$

$$\sqrt{49.5} = \frac{0.5}{14} + 7$$

$$= \frac{0.5+98}{14} = \frac{98.5}{14} = 7.036$$

Q.7. Show that the function *f* given by *f*(*x*) = $\tan^{-1}$ (sin *x* + cos *x*) is decreasing for all.

Ans.

We have

$$f(x) = \tan^{-1}(\sin x + \cos x)$$

$$\Rightarrow \quad f'(x) = \frac{1}{1+(\sin x+\cos x)^2} \times (\cos x - \sin x)$$

$$f'(x) = \frac{\cos x - \sin x}{1+(\sin x + \cos x)^2}$$

$$\because \quad 1 + (\sin x + \cos x)^2 > 0 \ \forall x \in R$$

Also, $\forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ $\sin x > \cos x$ $\quad \Rightarrow \cos x - \sin x < 0$

$\therefore \ f'(x) = \frac{-\text{ve}}{+\text{ve}} = -\text{ ve}$ $\quad$ *i.e.,* $f'(x) < 0$

$\Rightarrow f(x)$ is decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Q.8. The volume of a cube is increasing at the rate of 9 cm^3/s. How fast is its surface area increasing when the length of an edge is 10 cm?

Ans.

Let V and S be the volume and surface area of a cube of side x cm respectively.

Given $\frac{dV}{dt} = 9 \text{ cm}^3/\text{sec}$

We require $\left.\frac{dS}{dt}\right]_{x=10 \text{ cm}}$

Now $V = x^3$

$\Rightarrow \frac{dV}{dt} = 3x^2 . \frac{dx}{dt}$

$\Rightarrow 9 = 3x^2 . \frac{dx}{dt}$

$\Rightarrow \frac{dV}{dt} = \frac{9}{3x^2} = \frac{3}{x^2}$

Again, $\because S = 6x^2$ [By formula for surface area of a cube]

$\Rightarrow \frac{dS}{dt} = 12.x.\frac{dx}{dt}$

$= 12x.\frac{3}{x^2} = \frac{36}{x}$

$\Rightarrow \left.\frac{dS}{dt}\right]_{x=10 \text{ cm}} = \frac{36}{10} = 3.6 \text{ cm}^2/\text{sec.}$

Short Answer Questions (OIQ)

[2 Mark]

Q.1. The length *x* of a rectangle is decreasing at the rate of 3 cm/minute and the width *y* is increasing at the rate of 2 cm/minute, when *x* = 10 cm and *y* = 6 cm, find the rates of change of the perimeter.

Ans.

Let P be the perimeter of rectangle.

$P = 2(x + y)$

Differentiating w.r.t. 't' we get

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$\frac{dP}{dt} = 2(-3 + 2)$$

$$\left[\begin{array}{l}\text{Given} \\ \frac{dx}{dt} = -3\,\text{cm/min} \\ \frac{dy}{dt} = 2\,\text{cm/min}\end{array}\right.$$

$$\Rightarrow \quad \frac{dP}{dt} = 2 \times (-1)$$

$= -2$ cm/minute.

$\Rightarrow$ Perimeter of rectangle is decreasing at the rate of 2 cm/minute.

Q.2. A particle moves along the curve 6*y* = *x*^{3} + 2. Find the points on the curve at which the *y*-coordinate is changing 8 times as fast as the *x*-coordinate.

Ans.

Given $6y = x^3 + 2$ $\quad\quad\quad\quad ...(i)$

Differentiating w.r.t. t, we get

$$6\frac{dy}{dt} = 3x^2 . \frac{dx}{dt}$$

$$2\frac{dy}{dt} = x^2\frac{dx}{dt} \quad\quad\quad ...(ii)$$

From question $\frac{dy}{dt} = 8.\frac{dx}{dt}$

$\therefore (ii) \Rightarrow 2 \times 8\frac{dx}{dt} = x^2\frac{dx}{dt}$

$\Rightarrow x^2 = 16$ $\qquad\qquad \Rightarrow x = \pm 4$

If $x = 4, \;\; y = \frac{64+2}{6} = 11$

If $x = -4, \; y = \frac{-64+2}{6} = \frac{-62}{6} = \frac{-31}{3}$

Hence, the required points are $(4, 11)$ and $\left(-4, \frac{-31}{3}\right)$.

Q.3. The surface area of a spherical bubble is increasing at the rate of 2 cm^2/s. Find the rate at which the volume of the bubble is increasing at the instant if its radius is 6 cm.

Ans.

Let r be the radius of bubble, S the surface area, and V be the volume of bubble at time t.

Then $\frac{dS}{dt} = 2\,\text{cm}^2/\text{s}$ (given), $r = 6$ cm, $\frac{dV}{dt} = ?$

As $S = 4\pi r^2$ for spherical bubble.

$\therefore \quad \frac{dS}{dt} = \frac{d}{dt}(4\pi r^2) = 8\pi r\frac{dr}{dt}$

$\Rightarrow \quad 2\ \text{cm}^2/s = 8\pi r\frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{2}{8\pi r} = \frac{1}{4\pi r}\text{cm}/s$

Since, $\qquad V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\cdot 3\pi r^2\frac{dr}{dt} = 4\pi r^2 \cdot \frac{1}{4\pi r}\ \text{cm}^3/\text{s} = r\ cm^3/s = 6\ \text{cm}^3/\text{s}$
$[\because r = 6 \text{ cm given}]$

Hence, the volume of the bubble is increasing at the rate of 6 cm^3/s.

Q.4. Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.

Ans.

Slope of tangent to the given curve

$y = \sqrt{4x-3} - 1$ at $(x, y) = \frac{dy}{dx}$

$$\Rightarrow \quad \frac{2}{3} = \frac{1 \times 4}{2\sqrt{4x-3}} - 0$$

$$\Rightarrow \quad 4\sqrt{4x-3} = 3 \times 4$$

$$\Rightarrow \quad \sqrt{4x-3} = 3$$

$$\Rightarrow \quad 4x - 3 = 9$$

$$\Rightarrow \quad x = \frac{12}{4}$$

$$\Rightarrow \quad x = 3$$

If $x = 3$ then $y = \sqrt{4 \times 3 - 3} - 1 = 3 - 1 = 2$

Therefore, required point is $(3, 2)$.

Q.5. In a competition, a brave child tries to inflate a huge spherical balloon bearing slogans against child labour at the rate of 900 cubic centimeter of gas per second. Find the rate at which the radius of the balloon is increasing when its radius is 15 cm.

Ans.

$\frac{dr}{dt} = ?, \quad \text{when } r = 15 \text{ cm}$

$V = \frac{4}{3}\pi r^3$

Differentiating both sides with respect to '*t*', we get

$\frac{dV}{dt} = \frac{4}{3}\pi \, 3r^2 \frac{dr}{dt}$

$900 = 4\pi(15)^2 \frac{dr}{dt}$

$\Rightarrow \quad \frac{dr}{dt} = \frac{900}{225 \times 4 \times \pi} = \frac{1}{\pi} \text{cm} / \text{sec}$

Long Answer Questions-I (PYQ)

[4 Mark]

Q.1. The length of a rectangle is decreasing at the rate of 5 cm/min. and the width *y* is increasing at the rate of 4 cm/min. When *x* = 8 cm and *y* = 6 cm, find the change of (*a*) the perimeter (*b*) area of the rectangle.

Ans.

Given, $\frac{dx}{dt} = -5\,\text{cm/min}, \frac{dy}{dt} = 4\,\text{cm/min}$

Let x = length, and y = breadth

Perimeter of rectangle $P = 2(x + y)$

∴ Rate of change of P is

$$\frac{dP}{dt} = 2.\frac{dx}{dt} + 2\frac{dy}{dt}$$

$$\Rightarrow \frac{dP}{dt} = 2(-5) + 2(4) = -2$$

∴ Perimeter is decreasing at 2 m/s

If A be the area of rectangle then

$A = x \cdot y$

Differentiating w.r.t. 't', we get

$$\frac{dA}{dt} = x.\frac{dy}{dt} + y.\frac{dx}{dt}$$

$= x \times 4 + y \times (-5)$

$= 4x - 5y$

$$\therefore \left.\frac{dA}{dt}\right]_{\substack{x=8\\y=6}} = 4 \times 8 - 5 \times 6$$

$= 32 - 30$

$= 2\ \text{cm}^2/\text{min}.$

Q.2. Find the intervals in which the function is

$$f(\mathbf{x}) = \frac{\mathbf{3}}{\mathbf{2}}x^4 - 4x^3 - 45x^2 + 51$$

(a) strictly increasing

(b) strictly decreasing.

Ans.

Here, $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$

$\Rightarrow \quad f'(x) = 6x^3 - 12x^2 - 90x$

$\Rightarrow \quad f'(x) = 6x(x^2 - 2x - 15) = 6x(x+3)(x-5)$

Now for critical point $f'(x) = 0$

$6x(x+3)(x-5) = 0$

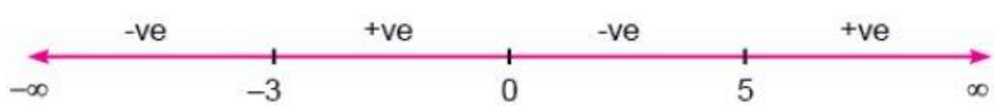
$x = 0, -3, 5$

i.e., $-3, 0, 5$ are critical points which divides domain R of given function into four disjoint sub intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 5)$, $(5, \infty)$.

For $(-\infty, -3)$

$f'(x) = +\text{ve} \times (-\text{ve}) \times (-\text{ve}) \times (-\text{ve}) = -\text{ve}$

i.e., $f(x)$ is decreasing in $(-\infty, -3)$



For $(-3, 0)$

$f'(x) = +\text{ve} \times (-\text{ve}) \times (+\text{ve}) \times (-\text{ve}) = +\text{ve}$

i.e., $f(x)$ is increasing in $(-3, 0)$

For $(0, 5)$

$f'(x) = +\text{ve} \times (+\text{ve}) \times (+\text{ve}) \times (-\text{ve}) = -\text{ve}$

i.e., $f(\text{x})$ is decreasing in $(0, 5)$

For $(5, \infty)$

$f'(x) = +ve \times (+ve) \times (+ve) \times (+ve) = +ve$

i.e., $f(x)$ is increasing in $(5, \infty)$

Hence $f(x)$ is

(a) strictly increasing in $(-3, 0) \cup (5, \infty)$

(b) strictly decreasing in $(-\infty, -3) \cup (0, 5)$

Q.3. Find the intervals in which *f*(*x*) = sin 3*x* – cos 3*x*, 0 < *x* < π, is strictly increasing or strictly decreasing.

Ans.

Given function is

$$f(x) = \sin 3x - \cos 3x$$
$$f'(x) = 3\cos 3x + 3\sin 3x$$

For critical points of function $f(x)$

$f'(x) = 0$

$\Rightarrow 3\cos 3x + 3\sin 3x = 0$

$\Rightarrow \cos 3x + \sin 3x = 0$

$\Rightarrow \sin 3x = - \cos 3x$

$\Rightarrow \frac{\sin 3x}{\cos 3x} = -1$

$\Rightarrow \tan 3x = -\tan \frac{\pi}{4}$

$\Rightarrow \tan 3x = \tan\left(\pi - \frac{\pi}{4}\right)$

$\Rightarrow \tan 3x = \tan \frac{3\pi}{4}$

$\Rightarrow 3x = n\pi + \frac{3\pi}{4}$

where $n = 0, \pm 1, \pm 2, ...$

Putting $n = 0, \pm 1, \pm 2, ...$ we get

$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12} \in (0, \pi)$

Hence required possible intervals are

$\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right) \left(\frac{11\pi}{12}, \pi\right)$

For $\left(0, \frac{\pi}{4}\right), f'(x) = +\text{Ve}$

For $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right), f'(x) = -\text{Ve}$

Hence, given function $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$ and strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$.

Q.4. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Ans.

Given, curve $ay^2 = x^3$

On differentiating, we get

$$2ay\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\Rightarrow \quad \frac{dy}{dx} \text{ at } (am^2, am^3) = \frac{3 \times a^2m^4}{2a \times am^3} = \frac{3m}{2}$$

$$\therefore \text{Slope of normal} = -\frac{1}{\text{slope of tangent}} = -\frac{1}{\frac{3m}{2}} = -\frac{2}{3m}$$

Equation of normal at the point (am^2, am^3) is given by

$$\frac{y-am^3}{x-am^2} = -\frac{2}{3m}$$

$$\Rightarrow \quad 3my - 3am^4 = -2x + 2am^2$$

$$\Rightarrow 2x + 3my - am^2(2 + 3m^2) = 0$$

Hence, equation of normal is $2x + 3my - am^2(2 + 3m^2) = 0$

Q.5. Find the approximate value of *f* (3.02), upto 2 places of decimal, where *f* (*x*) = $3x^2 + 5x + 3$.

Ans.

Here, $f(x) = 3x^2 + 5x + 3$

Let $x = 3$ and $dx = 0.02$

$\therefore x + dx = 3.02$

By definition, the approximate value of $f(x)$ is

$$f'(x) = \frac{f(x+dx)-f(x)}{dx}$$
$$\Rightarrow \quad f'(3) = \frac{f(3+0.02)-f(3)}{0.02}$$

[Putting $x = 3$ and $dx = 0.02$]

$$\Rightarrow f'(x) = \frac{f(x+dx)-f(x)}{dx} \quad ...(i)$$
$$\Rightarrow \quad f'(3) = \frac{f(3+0.02)-f(3)}{0.02}$$

Now, $f(x) = 3x^2 + 5x + 3$

$\Rightarrow f'(x) = 6x + 5$

$\Rightarrow f'(3) = 23$

Also $f(3) = 3 \times 3^2 + 5 \times 3 + 3 = 27 + 15 + 3 = 45$

Putting in (i), we get

$$23 = \frac{f(3.02)-45}{0.02}$$

$\Rightarrow f(3.02) = 23 \times 0.02 + 45 = 45.46$

Q.6. Show that $y = \log(1+x) - \frac{2x}{2+x}, x >-$ **is an increasing function of *x* throughout its domain.**

Here, $f(x) = \log\,(1 + x) - \frac{2x}{2+x}$

[where $y = f\,(x)$]

$$\Rightarrow \; f'(x) = \frac{1}{1+x} - 2\left[\frac{(2+x).1 - x}{(2+x)^2}\right]$$

$$= \frac{1}{1+x} - \frac{2(2+x - x)}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{4+x^2+4x-4-4x}{(x-1)\,(x+2)^2}$$

$$= \frac{x^2}{(x+1)\,(x+2)^2}$$

For $f(x)$ being increasing function

$f'(x) > 0$

$$\Rightarrow \quad \frac{x^2}{(x+1)\,(x+2)^2} 0$$

$$\Rightarrow \quad \frac{1}{x+1} \cdot \frac{x^2}{(x+2)^2} 0$$

$$\Rightarrow \quad \frac{1}{x+1} 0 \qquad \left[\frac{x^2}{(x+2)^2} 0\right]$$

$\Rightarrow \; x + 1 > 0 \quad$ or $\quad x > -1$

i.e., $f(x) = y = \log\,(1 + x) - \frac{2x}{2+x}$ is increasing function in its domain $x > -1$ *i.e.,* $(-1, \infty)$.

Q.7. Find the equation of tangent to the curve
$x = \sin 3t,\ y = \cos 2t$ at $t = \frac{\pi}{4}$.

Ans.

Here, $y = \cos 2t \quad \therefore \quad \frac{dy}{dt} = -2\sin 2t$

Also, $x = \sin 3t \quad \therefore \quad \frac{dx}{dt} = 3\cos 3t$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2 \sin 2t}{3 \cos 3t}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{-2 \sin\frac{\pi}{2}}{3\cos \frac{3\pi}{4}}$$

$$= \frac{-2\times1}{3\times\left(-\frac{1}{\sqrt{2}}\right)} = \frac{2\sqrt{2}}{3}$$

If $t = \frac{\pi}{4}$ then

$$x = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}},$$
$$y = \cos\frac{2\pi}{4} = \cos\frac{\pi}{2} = 0$$

Therefore, equation of tangent at $t = \frac{\pi}{4} i.e.$, at $\left(\frac{1}{\sqrt{2}}, 0\right)$ is given by

$$y - 0 = \frac{dy}{dx}\left(x - \frac{1}{\sqrt{2}}\right)$$
$$\Rightarrow \quad y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad 3y = 2\sqrt{2x} - 2$$

Q.8. Using differential, find the approximate value of $f(2.01)$, where $f(x) = 4x^3 + 5x^2 + 2$.

Ans.

Let $x = 2, \Delta x = 0.01$ where $f(x) = 4x^3 + 5x^2 + 2$

$\Rightarrow \qquad x + \Delta x = 2 + 0.01 = 2.01$

By definition, approximate value of $f(x)$ is

$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$\left[\because f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$

$\Rightarrow f'(2) = \frac{f(2+0.01) - f(2)}{0.01} \qquad \ldots(i)$

$\because \quad f(x) = 4x^3 + 5x^2 + 2$

$\Rightarrow \quad f'(x) = 12x^2 + 10x$

$\Rightarrow \quad f'(2) = 48 + 20 = 68$

Also, $f(2) = 4 \times 2^3 + 5 \times 2^2 + 2$

$= 32 + 20 + 2 = 54$

Putting the values of $f'(2)$ and $f(2)$ in (i) we get

$68 = \frac{f(2.01) - 54}{0.01}$

$\Rightarrow f(2.01) = 68 \times 0.01 + 54 = 54.68$

Q.9. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the *y*-coordinate of the point.

Ans.

Let $P(x_1, y_1)$ be the required point on the curve

$y = x^3$ ...(i)

$\frac{dy}{dx} = 3x^2$

$\Rightarrow \left[\frac{dy}{dx}\right]_{(x_1,y_1)} = 3{x_1}^2$

$\Rightarrow$ Slope of tangent at $(x_1, y_1) = 3x_1^2$

According to the question,

$3x_1^2 = y_1$ ...(ii)

Also (x_1, y_1) lies on (i)

$\Rightarrow y_1 = x_1^3$ ...(iii)

From (ii) and (iii), we get

$3x_1^2 = x_1^3$

$\Rightarrow x_1^3 - 3x_1^2 = 0$

$\Rightarrow x_1^2 (x_1 - 3) = 0$

$\Rightarrow x_1 = 0$ or $x_1 = 3$

$\Rightarrow y_1 = 0$ or $y_1 = 27$

Hence, required points are $(0, 0)$ and $(3, 27)$.

Q.10. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm?

Ans.

Let 'A' be the area and 'a' be the side of an equilateral triangle.

$$A = \frac{\sqrt{3}}{4} a^2$$

Differentiating with respect to t we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2a \cdot \frac{da}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2a \times 2 \quad \left[\text{Given } \frac{da}{dt} = 2 \text{ cm/sec}\right]$$

$$\frac{dA}{dt} = \sqrt{3a}$$

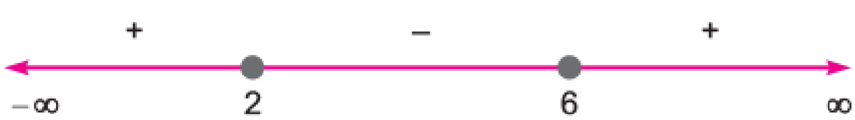
$$\Rightarrow \quad \left.\frac{dA}{dt}\right]_{a=20\,\text{cm}} = 20\sqrt{3}\,\text{sq cm/s}$$

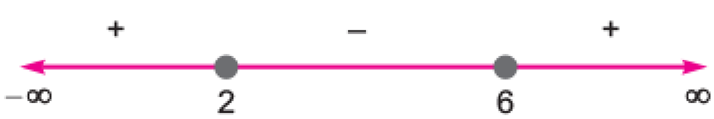
Long Answer Questions-I (OIQ)

[4 Mark]

Q.1. Find the intervals in which the function $f(x) = x^3 - 12x^2 + 36x + 17$ is (a) increasing, (b) decreasing.

Ans.





We have, $f(x) = x^3 - 12x^2 + 36x + 17$

$\Rightarrow f(x) = 3x^2 - 24x + 36 = 3\ (x - 6)\ (x - 2)$

a. For $f(x)$ to be increasing, we must have

$f'(x) > 0$

$\Rightarrow 3\ (x - 6)\ (x - 2) > 0$

$\Rightarrow x < 2$ or $x > 6$

$\Rightarrow x \in (-\infty, 2) \cup (6, \infty)$

So, $f(x)$ is increasing on $(-\infty, 2) \cup (6, \infty)$

b. For $f(x)$ to be decreasing, we must have

$f'(x) < 0$

$\Rightarrow 3(x - 2)(x - 6) < 0$

$\Rightarrow 2 < x < 6$

So, $f(x)$ is decreasing on $(2, 6)$.

Q.2. Find the equation of tangent to the curve $y = \sqrt{3x - 2}$, which is parallel to the line 4*x* – 2*y* + 5 = 0.

Ans.

Given, curve $y = \sqrt{3x - 2} \quad \Rightarrow \quad y^2 = 3x - 2$ ...(*i*)

To get the equation of tangent to the curve $y^2 = 3x - 2$, which is a parabola, first we have to find the coordinates of point from where tangent line passes.

Let the coordinates of the point on parabola be (a, b), then this coordinate will satisfy the equation (i).

Therefore, from (i), we have $b^2 = 3a - 2$ ...(*ii*)

Now differentiating the equation (i) with respect to *x*, we get

$$2y\frac{\mathrm{dy}}{\mathrm{dx}} = 3$$
$$\Rightarrow \qquad \left(\frac{\mathrm{dy}}{\mathrm{dx}}\right) = \frac{3}{2y}$$

Now slope of the required tangent line

$$m_1 = \left(\frac{dy}{dx}\right)_{(a,b)} = \frac{3}{2b}$$

As it is given that required tangent line is parallel to the given line $4x - 2y + 5 = 0$, so, the slopes of lines are equal.

Therefore,

$$\frac{3}{2b} = \frac{-4}{-2}$$
$$\Rightarrow 4b = 3$$
$$\Rightarrow \quad b = \frac{3}{4}$$

Substituting this value $b = \frac{3}{4}$ in equation (ii), we get

$$\left(\frac{3}{4}\right)^2 + 2 = 3a$$
$$\Rightarrow \qquad 3a = \frac{41}{16} \qquad \Rightarrow \qquad a = \frac{41}{48}$$

Now the coordinates of the point on tangent are $\frac{41}{48}, \frac{3}{4}$ and slope is 2.

Hence, equation of tangent is obtained by $y - b = m(x - a)$

$$\Rightarrow \quad y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$$
$$\Rightarrow \quad \frac{4y - 3}{4} = \frac{2(48x - 41)}{48}$$
$$\Rightarrow \quad (4y - 3) = \frac{48x - 41}{6}$$
$$\Rightarrow \quad 24y - 18 = 48x - 41$$

$\Rightarrow 48x - 24y = 41 - 18$ $\qquad$ $\Rightarrow 48x - 24y - 23 = 0$ is the required equation of tangent.

Q.3. The fuel cost for running a train is proportional to the square of the speed generated in km per hour. If the fuel costs ₹ 48 per hour at speed 16 km per hour and the fixed charges amount to ₹ 1200 per hour then find the most economical speed of train, when total distance covered by train is *S* km.

Ans.

Let F be the fuel cost per hour and v the speed in km/h

From question,

$F \propto v^2 \qquad \Rightarrow F = Kv^2 \qquad \ldots(i)$

Given, $F =$ ₹ 48/hour, when $v = 16$ km/hour

$\Rightarrow 48 = K.16^2 \qquad \Rightarrow \qquad K = \frac{3}{16}$

Now (i) becomes, $F = \frac{3v^2}{16}$

If t is the time taken by train in covering given distance S km and C the total cost for running the train then

$$\Rightarrow C = 1200\,t + \frac{3v^2}{16}t \qquad \left[\because\ t = \frac{S}{v}\right]$$

$$\Rightarrow C = 1200 \times \frac{S}{v} + \frac{3v^2}{16} \times \frac{S}{v}$$

$$\Rightarrow \frac{dC}{dv} = -\,1200 \times \frac{S}{v^2} + \frac{3S}{16}$$

For maximum or minimum value of $C, \frac{dC}{dv} = 0$

$$\Rightarrow \qquad -1200 \times \frac{S}{v^2} + \frac{3S}{16} = 0$$

$$\Rightarrow \qquad 1200 \times \frac{S}{v^2} = \frac{3S}{16}$$

$$\Rightarrow \qquad v^2 = \frac{1200 \times 16}{3}$$

$$\Rightarrow \qquad v^2 = 6400 \qquad \Rightarrow \qquad v = 80\ \text{km}/\text{hour}$$

Also,

$$\Rightarrow \quad \frac{d^2C}{dv^2} = 2400 \times \frac{S}{v^3}$$

$$\Rightarrow \quad \left(\frac{d^2C}{dv^2}\right)_{v=80} > 0$$

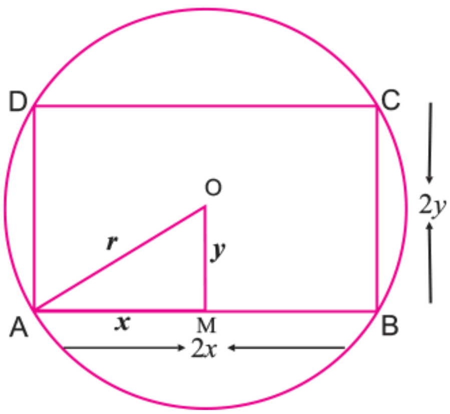
Hence, C is minimum, when $v = 50$ km/hour.

Long Answer Questions-II (PYQ)

[6 Marks]

Q.1. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius r is the square of side $r\sqrt{2}$.

Ans.



Let $ABCD$ be a rectangle inscribed in a circle of radius r with centre at O. Let $AB = 2x$ and

$BC = 2y$ be the sides of the rectangle.

Then in right angled ΔOAM, $AM^2 + OM^2 = OA^2$ (By Pythagoras theorem)

$$\Rightarrow x^2 + y^2 = r^2 \quad \Rightarrow \quad y = \sqrt{r^2 - x^2} \qquad ...(i)$$

Let P be the perimeter of rectangle $ABCD$, then

$$P = 4x + 4y \Rightarrow P = 4x + 4\sqrt{r^2 - x^2}$$

$$\Rightarrow \quad \frac{dP}{dx} = 4 - \frac{4x}{\sqrt{r^2 - x^2}}$$

For maximum or minimum value of P, we have $\frac{dP}{dx} = 0$

$$4 - \frac{4x}{\sqrt{r^2 - x^2}} = 0$$

$$\Rightarrow \quad 4 = \frac{4x}{\sqrt{r^2 - x^2}}$$

$$\sqrt{r^2 - x^2} = x \quad \Rightarrow \quad r^2 - x^2$$

$$\Rightarrow \quad 2x^2 = r^2 \quad \Rightarrow \quad x = \frac{r}{\sqrt{2}}$$

Now $$\frac{d^2P}{dx^2} = \frac{-4\left\{\sqrt{r^2-x^2} - \frac{x.(-x)}{\sqrt{r^2-x^2}}\right\}}{(\sqrt{r^2-x^2})^2}$$

$$= -\frac{4r^2}{(r^2-x^2)^{3/2}}$$

$$\therefore \left(\frac{d^2P}{dx^2}\right)_{x=\frac{r}{\sqrt{2}}} = \frac{-4r^2}{\left(r^2-\frac{r^2}{2}\right)^{3/2}} = \frac{-8\sqrt{2}}{r} < 0$$

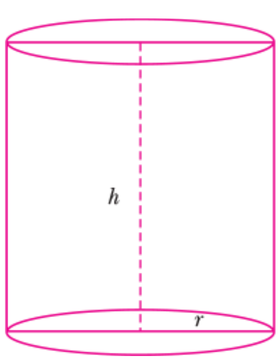
Thus, P is maximum when $x = \frac{r}{\sqrt{2}}$. Putting $x = \frac{r}{\sqrt{2}}$ in (i), we get $y = \frac{r}{\sqrt{2}}$

Therefore, $x = y \quad \Rightarrow 2x = 2y$

Hence, P is maximum when rectangle is a square of side $2x = \sqrt{2}r$.

Q.2. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area.

Ans.



Let r, h be radius and height of closed right circular cylinder having volume 128π cm^3.

If S be the surface area then

$$S = 2\pi rh + 2\pi r^2 \qquad \Rightarrow S = 2\pi(rh + r^2)$$

$$S = 2\pi\left(r.\frac{128}{r^2} + r^2\right) \qquad \left[\begin{array}{l}\because V = \pi r^2 h \\ \Rightarrow\ 128\pi = \pi r^2 h \\ \therefore\ h = \frac{128}{r^2}\end{array}\right]$$

$$S = 2\pi\left(\frac{128}{r} + r^2\right)$$

$$\Rightarrow \quad \frac{dS}{dr} = 2\pi\left(-\frac{128}{r^2} + 2r\right)$$

For extreme value of S

$$\frac{dS}{dr} = 0 \quad \Rightarrow \quad 2\pi\left(-\frac{128}{r^2} + 2r\right) = 0$$

$$\Rightarrow \qquad -\frac{128}{r^2} + 2r = 0$$

$$\Rightarrow \qquad 2r = \frac{128}{r^2}$$

$$\Rightarrow \qquad r^3 = \frac{128}{2}$$

$$\Rightarrow r^3 = 64 \quad \Rightarrow r = 4$$

Again

$$\frac{d^2S}{dr^2} = 2\pi\left(\frac{128\times2}{r^3} + 2\right)$$

$$\Rightarrow \quad \left.\frac{d^2S}{dr^2}\right]_{r=4} = +\text{ve}$$

Hence, for $r = 4$ cm, S (surface area) is minimum.

Therefore, dimensions for minimum surface area of cylindrical can are

radius $r = 4$ cm and $h = \frac{128}{r^2} = \frac{128}{16} = 8\,\text{cm}$

Q.3. Prove that the surface area of a solid cuboid, of square base and given volume, is minimum when it is a cube.

Ans.

Let x be the side of square base of cuboid and other side be y.

Then volume of cuboid with square base, $V = x\,.\,x\,.\,y = x^2y$

As volume of cuboid is given so volume is taken constant throughout the question, therefore,

$$y = \frac{V}{x^2} \qquad ...(i)$$

In order to show that surface area is minimum when the given cuboid is cube, we have to show S″ > 0 and $x = y$.

Let S be the surface area of cuboid, then

$S = x^2 + xy + xy + xy + xy + x^2$

$$S = 2x^2 + 4xy \qquad ...(ii)$$

$\Rightarrow \quad S = 2x^2 + 4x.\frac{V}{x^2}$

$\Rightarrow \quad S = 2x^2 + \frac{4V}{x} \quad ...(iii)$

$\Rightarrow \quad \frac{dS}{dx} = 4x - \frac{4V}{x^2} \quad ...(iv)$

For maximum/minimum value of S, we have $\frac{dS}{dx} = 0$

$\Rightarrow 4x - \frac{4V}{x^2} = 0 \Rightarrow 4V = 4x^3$

$\Rightarrow V = x^3 \quad ...(v)$

Putting $V = x^3$ in (i), we have

$y = \frac{x^3}{x^2} = x$

Here, $y = x \Rightarrow$ cuboid is a cube.

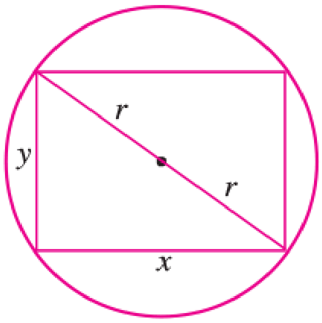
Differentiating (iv) w.r.t x, we get

$\frac{d^2S}{dx^2} = \left(4 + \frac{8V}{x^3}\right) > 0$

Hence, surface area is minimum when given cuboid is a cube.

Q.4. Show that the rectangle of maximum area that can be inscribed in a circle is a square.

Ans.



Let x and y be the length and breadth of a rectangle inscribed in a circle of radius r.

Then $y^2 + x^2 = 4r^2$

$\Rightarrow \quad y = \sqrt{4r^2 - x^2}$ ...(i)

If A be the area of rectangle then $A = x \cdot y$

$$A = x.\sqrt{4r^2 - x^2} \text{ (using } (i))$$

$$\Rightarrow \quad \frac{dA}{dx} = x.\frac{1}{2\sqrt{4r^2 - x^2}}(-2x) + \sqrt{4r^2 - x^2}$$

$$= -\frac{x^2}{\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2} = \frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

For maxima or minima,

$$\frac{dA}{dx} = 0 \quad \Rightarrow \quad \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0$$

$$\Rightarrow \quad 4r^2 - 2x^2 = 0$$

$$\Rightarrow 2x^2 = 4r^2 \Rightarrow \quad x = \sqrt{2}r$$

Now,

$$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}(-4x) - \frac{(4r^2 - 2x^2)(-2x)}{2\sqrt{4r^2 - x^2}}}{(4r^2 - x^2)}$$

$$= \frac{2x(x^2 - 6r^2)}{(4r^2 - x^2)^{3/2}}$$

$$\therefore \left[\frac{d^2A}{dx^2}\right]_{x=\sqrt{2}r} = \frac{2\sqrt{2}r\,(2r^2-4r^2)}{(4r^2-2r^2)^{3/2}}$$

$$= \frac{-4\sqrt{2}r^3}{(2r^2)^{3/2}} < 0$$

Hence, A is maximum when $x = \sqrt{2}r$

Putting $x = \sqrt{2}r$ in (i) we get

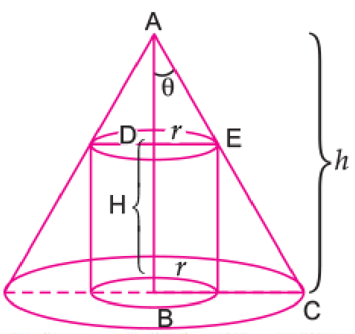
$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2}r$$

i.e., $x = y = \sqrt{2}r$

Therefore, area of rectangle is maximum when $x = y$ *i.e.,* rectangle is a square.

Q.5. Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}$ h

Ans.



Let r and H be the radius and height of inscribed cylinder respectively and θ be the semi-vertical angle of given cone.

If V be the volume of cylinder.

then $V = \pi r^2 H$

$\therefore V = \pi r^2 \,(h - r\cot\theta)$

$\Rightarrow V = \pi(hr^2 - r^3\cot\theta)$

Differentiating with respect to r, we get

$$\begin{bmatrix} \text{In } \Delta ADE \\ \cot\theta = \frac{AD}{DE} \\ \cot\theta = \frac{h-H}{r} \\ \therefore\ H = h - r\cot\theta \end{bmatrix}$$

$$\Rightarrow \frac{dV}{dr} = \pi\left(2rh - 3r^2\cot\theta\right)$$

For maxima or minima,

$$\frac{dV}{dr} = 0$$

$$\Rightarrow \pi\left(2rh - 3r^2\cot\theta\right) = 0 \qquad \Rightarrow 2rh - 3r^2\cot\theta = 0 \qquad [\therefore \pi \neq 0]$$

$$\Rightarrow r\left(2h - 3r\cot\theta\right) = 0$$

$$2h - 3r\cot\theta = 0$$

$$\Rightarrow \quad r = \frac{2h}{3\cot\theta} = \frac{2h}{3}\tan\theta \qquad \Rightarrow [\because r \neq 0]$$

Now, $\frac{d^2V}{dr^2} = \pi\left(2h - 6r\cot\theta\right)$

$$\left.\frac{d^2V}{dr^2}\right]_{r=\frac{2h}{3}\tan\theta} = \pi\left(2h - 6 \times \frac{2h}{3}\tan\theta.\ \cot\theta\right)$$

$$= \pi(2h - 4h) < 0$$

Hence, volume will be maximum when $r = \frac{2h}{3}\tan\theta$

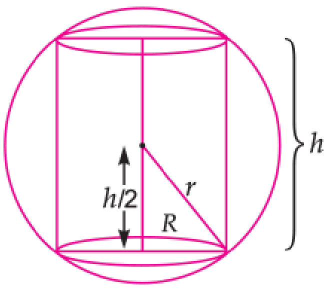
$\therefore H$ (height of cylinder) $= h - \frac{2h}{3}\tan\theta.\ \cot\theta = \frac{3h-2h}{3} = \frac{h}{3}$

Q.6. Find the volume of the largest cylinder that can be inscribed in a sphere of radius *r*.

OR

Show that the height of the cylinder of maximum volume, that can be inscribed in a sphere of radius *R* is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Ans.



Let R, h be the radius and height of inscribed cylinder respectively.

If V be the volume of cylinder then

$$V = \pi R^2 h$$

$$\left[\begin{array}{l} \because\ R^2 + \left(\frac{h}{2}\right)^2 = r^2 \\ R^2 = r^2 - \frac{h^2}{4} \end{array}\right.$$

$$V = \pi\left(r^2 - \frac{h^2}{4}\right)h$$

$$V = \pi\left(r^2 h - \frac{h^3}{4}\right)$$

Differentiating with respect to h, we get

$$\frac{dV}{dh} = \pi\left(r^2 - \frac{3h^2}{4}\right) \qquad \ldots(i)$$

For maxima or minima

$$\frac{dV}{dh} = 0$$

$$\Rightarrow \pi\left(r^2 - \frac{3h^2}{4}\right) = 0 \qquad \Rightarrow r^2 - \frac{3h^2}{4} = 0$$

$$\Rightarrow r = \frac{h\sqrt{3}}{2} \qquad \Rightarrow h = \frac{2r}{\sqrt{3}}$$

Differentiating (i) again with respect to h, we get

$$\Rightarrow \frac{d^2V}{dh^2} = -\frac{\pi 6h}{4} \qquad \Rightarrow \left.\frac{d^2V}{dh^2}\right]_{h=\frac{2r}{\sqrt{3}}} = -\frac{3\pi}{2}.\frac{2r}{\sqrt{3}} < 0$$

Hence, V is maximum when $h = \frac{2r}{\sqrt{3}}$.

$$\therefore \text{Maximum volume} = \pi\left(r^2.\frac{2r}{\sqrt{3}} - \frac{8r^3}{4\times 3\sqrt{3}}\right)$$

$$= \pi\left(\frac{24r^3 - 8r^3}{12\sqrt{3}}\right)$$

$$= \pi\frac{16r^3}{12\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}$$

Q.7. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth in 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per sq. metre for the base and ₹ 45 per sq. metre for sides, what is the cost of least expensive tank?

Ans.

Let l and b be the length and breadth of the tank.

If C be the cost of constructing the tank then

$C = 70\ lb + 45 \times 2\ (2l + 2b)$ [depth = $2m$; area of 4 sides = $2(l \times 2 + b \times 2)$]

$= 70\ lb + 180\ l + 180\ b$

$$C = 70l \times \frac{4}{l} + 180l + 180 \times \frac{4}{l}$$

$$\Rightarrow \quad C = 280 + 180\left(l + \frac{4}{l}\right)$$

Differentiating with respect to l, we get

$$\frac{dC}{dl} = 180\left(1 - \frac{4}{l^2}\right) \qquad \ldots(i)$$

For maxima or minima

$$\frac{dC}{dl} = 0$$

$$\Rightarrow 180\left(1 - \frac{4}{l^2}\right) = 0 \qquad \Rightarrow l_2 = 4 \qquad \Rightarrow l = 2 \qquad [\because l \neq -2]$$

Differentiating (i) again with respect to l, we get

$$\frac{d^2C}{dl^2} = 180 + \frac{8}{l^3}$$

$$\Rightarrow \left.\frac{d^2C}{dl^2}\right]_{l=2} = 181 > 0 \qquad [\because l = -2]$$

Here C is minimum when $l = 2$

$$\therefore b = \frac{4}{2} = 2$$

Minimum $\text{cost} = 280 + 180\left(2 + \frac{4}{2}\right) = 280 + 720 =$ ₹1000.

Q.8. If the sum of hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is $-\frac{\pi}{3}$.

Ans.

Let h and x be the length of hypotenuse and one side of a right triangle and y is length of the third side.

If A be the area of triangle, then

$$A = \frac{1}{2}\text{xy} = \frac{1}{2}x\sqrt{h^2 - x^2}$$

$$\begin{bmatrix} \text{also given} \\ h + x = k \text{ (constant)} \\ \therefore \quad h = k - x \end{bmatrix}$$

$$A = \frac{1}{2}x\sqrt{(k-x)^2 - x^2} = \frac{1}{2}x\sqrt{k^2 - 2\text{kx} + x^2 - x^2}$$

$$A^2 = \frac{x^2}{4}(k^2 - 2\,\text{kx})$$

$$\Rightarrow \quad A^2 = \frac{1}{4}(k^2x^2 - 2\,\text{kx}^3)$$

Differentiating with respect to x we get

$$\frac{d(A^2)}{dx} = \frac{1}{4}(2k^2x - 6kx^2) \qquad \ldots(i)$$

For maxima or minima of A^2

$$\frac{d(A^2)}{dx} = 0 \quad \Rightarrow \quad \frac{1}{4}(2k^2x - 6kx^2) = 0$$

$$\Rightarrow 2k^2x - 6kx^2 = 0 \qquad \Rightarrow \quad 2kx\,(k - 3x) = 0$$

$\Rightarrow k - 3x = 0 ;$ $\qquad 2kx \neq 0$

$$\begin{bmatrix} \because V = lbh \\ 8 = lb2 \\ \therefore\ b = \frac{8}{2l} = \frac{4}{l} \end{bmatrix}$$

$\Rightarrow \quad x = \frac{k}{3}$

Differentiating (i) again with respect to x, we get

$$\frac{d^2(A^2)}{dx^2} = \frac{1}{4}(2k^2 - 12\,kx)$$

$$\left.\frac{d^2(A^2)}{dx^2}\right]_{x=k/3} = \frac{1}{4}\left(2k^2 - 12k.\frac{k}{3}\right)0$$

Hence, A^2 is maximum when $x = \frac{k}{3}$ and $h = k - \frac{k}{3} = \frac{2k}{3}$.

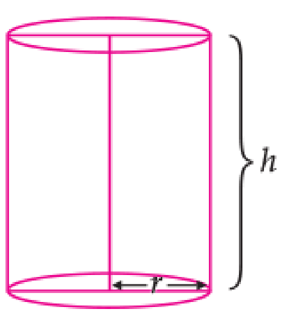
i.e., A is maximum when $x = \frac{k}{3}, h = \frac{2k}{3}$

$\therefore\ \cos\theta = \frac{x}{h} = \frac{k}{3} \times \frac{3}{2k} = \frac{1}{2}$

$\Rightarrow \quad \cos\theta = \frac{1}{2} \qquad \Longrightarrow \qquad \theta = \frac{\pi}{3}$

Q.9. Show that the right circular cylinder, open at the top, and of given surface area and maximum volume is such that its height is equal to the radius of the base.

Ans.



Let r, h be the radius and height of given cylinder respectively, having surface area S.

If V be the volume of cylinder, then

$V = \pi r^2 h$

$\therefore \; V = \pi r^2 . \left(\frac{S - \pi r^2}{2\pi r}\right)$

$\Rightarrow V = \frac{Sr - \pi r^3}{2}$

$\frac{dV}{dr} = \frac{1}{2}(S - 3\pi r^2)$

$\left[\begin{array}{l} S = \pi r^2 + 2\pi rh \;\; ...(i) \\ \therefore h = \dfrac{S - \pi r^2}{2\pi r} \end{array}\right]$

For maxima or minima of V,

$\frac{dV}{dr} = 0 \quad \Rightarrow \quad \frac{1}{2}(S - 3\pi r^2) = 0 \quad \Rightarrow \quad r = \sqrt{\frac{S}{3\pi}}$

Now, $\frac{d^2V}{dr^2} = \frac{1}{2}(-6\pi r) \quad \Rightarrow \left.\frac{d^2V}{dr^2}\right]_{r=\sqrt{\frac{S}{3\pi}}} < 0$

V is maximum when $r = \sqrt{\frac{S}{3\pi}} \quad \Rightarrow \quad 3\pi r^2 = S$

Putting it in (i)

$3\pi r^2 = \pi r^2 + 2\pi rh \quad \Rightarrow 2\pi r^2 = 2\pi rh \quad \Rightarrow r = h$

i.e., V is maximum when $r = h$ *i.e.*, height is equal to the radius of base.

Q.10. The length of the sides of an isosceles triangle are $9 + x^2$, $9 + x^2$ and $18 - 2x^2$ units. Calculate the area of the triangle in terms of x and find the value of x which makes the area maximum.

Ans.

Sides of isosceles triangles are $9 + x^2$, $9 + x^2$ and $18 - 2x^2$

$$S = \frac{9+x^2+9+x^2+18-2x^2}{2} = \frac{36}{2} = 18$$

If A be area of triangle, then

$$A = \sqrt{S\ (S-a)\ (S-b)\ (S-c)}$$

$$A = \sqrt{18(18-9-x^2)\ (18-9-x^2)\ \ (18-18+2x^2)}$$

$$A = \sqrt{18\ (9-x^2)\ (9-x^2)\ \cdot 2x^2}$$

$$A = 6x\ (9-x^2) = 6\ (9x-x^3)$$

For maxima or minima of A

$\frac{dA}{dx} = 6\,(9-3x^2) = 0 \qquad \Rightarrow 9-3x^2 = 0 \quad \Rightarrow \quad x = \pm\sqrt{3}$

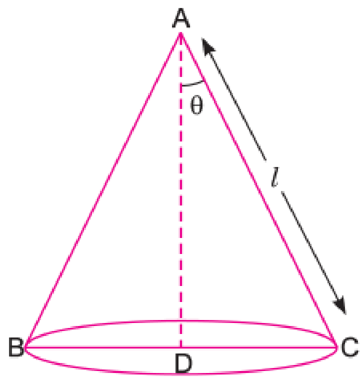
Again, $\frac{d^2A}{dx^2} = 6\,(-6x) = -36x$

Now, $\left.\frac{d^2A}{dx^2}\right]_{x=\sqrt{3}} = -\ 36\sqrt{3} < 0$ and $\left.\frac{d^2A}{dx^2}\right]_{x=-\sqrt{3}} = -36\ (-\sqrt{3}) > 0$

Hence, for $x = \sqrt{3}$, Area (A) is maximum.

Q.11. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is .

Ans.



Let *ABC* be cone having slant height *l* and semi-vertical angle θ.

If *V* be the volume of cone then.

$V = \frac{1}{3}.\pi \times DC^2 \times AD \ = \frac{\pi}{3} \times l^2 \sin^2\theta \times l\cos\theta$

$\Rightarrow \qquad V = \frac{\pi l^3}{3}\sin^2\theta\cos\theta$

$\Rightarrow \qquad \frac{dV}{d\theta} = \frac{\pi l^3}{3}[-\sin^3\theta + 2\sin\theta.\cos^2\theta]$

For maximum value of *V*.

$\Rightarrow \quad \frac{dV}{d\theta} = 0$

$\Rightarrow \quad \frac{\pi l^3}{3}\ [-\sin^3\theta + 2\sin\theta.\cos^2\theta] = 0$

$\Rightarrow\ -\sin^3\theta + 2\sin\theta.\cos^2\theta = 0$

$\Rightarrow -\sin\theta\ (\sin^2\theta - 2\cos^2\theta) = 0$

$\Rightarrow \sin\theta = 0$ or $1 - \cos^2 q\ - 2\cos^2\theta = 0$

$\Rightarrow \theta = 0$ or $1 - 3\cos^2\theta = 0$

$\Rightarrow \theta = 0$ or $\cos\theta = \frac{1}{\sqrt{3}}$

Now $\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\{-3\sin^2\theta.\cos\theta - 4\sin^2\theta.\cos\theta + 2\cos^3\theta\}$

$\Rightarrow \frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\{-7\sin^2\theta\cos\theta + 2\cos^3\theta\}$

$\Rightarrow \left.\frac{d^2V}{d\theta^2}\right]_{\theta=0} = +\text{ve}$

and

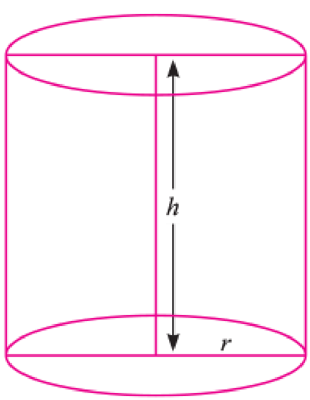
$\left.\frac{d^2V}{d\theta^2}\right]_{\cos\theta=\frac{1}{\sqrt{3}}} = -\text{ve}$ [Putting $\cos\theta$

$= \frac{1}{\sqrt{3}}$ and $\sin\theta = \sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\sqrt{2}}{\sqrt{3}}$]

Hence, for $\cos\theta = \frac{1}{\sqrt{3}}$ or $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, V is maximum.

Q.12. Show that the height of a closed right circular cylinder of given surface and maximum volume, is equal to the diameter of its base.

Ans.



Let r and h be radius and height of given cylinder of surface area S.

If V be the volume of cylinder then

$V = \pi r^2 h$

$$V = \frac{\pi r^2.(S - 2\pi r^2)}{2\pi r}$$

$$[\because \quad S = 2\pi r^2 + 2\pi \mathrm{rh} \quad \Rightarrow \quad \frac{S - 2\pi r^2}{2\pi r} = h]$$

$$V = \frac{\mathrm{Sr} - 2\pi r^3}{2}$$

$$\Rightarrow \quad \frac{\mathrm{dV}}{\mathrm{dr}} = \frac{1}{2} \quad (S - 6\pi r^2)$$

For maximum or minimum value of V

$$\frac{\mathrm{dV}}{\mathrm{dr}} = 0$$

$$\Rightarrow \frac{1}{2}(S - 6\pi r^2) = 0$$

$$\Rightarrow S - 6\pi r^2 = 0$$

$$\Rightarrow r^2 = \frac{S}{6\pi}$$

$$\Rightarrow r = \sqrt{\frac{S}{6\pi}}$$

Now $\frac{d^2V}{dr^2} = -\frac{1}{2} \times 12\pi r \quad \Rightarrow \frac{d^2V}{dr^2} = -6\pi r \Rightarrow \left[\frac{d^2V}{dr^2}\right]_{r=\sqrt{\frac{S}{6\pi}}} < 0$

Hence, for $r = \sqrt{\frac{S}{6\pi}}$, volume V is maximum.

$$\Rightarrow h = \frac{S - 2\pi.\frac{S}{6\pi}}{2\pi\sqrt{\frac{S}{6\pi}}} \quad \Rightarrow \quad h = \frac{3S - S}{3\times 2\pi} \times \sqrt{\frac{6\pi}{S}}$$

$$\Rightarrow h = \frac{2S}{6\pi}.\frac{\sqrt{6\pi}}{\sqrt{S}} = 2\sqrt{\frac{S}{6\pi}}$$

$$\Rightarrow h = 2r \text{ (diameter)} \qquad \left[\because r = \sqrt{\frac{S}{6\pi}}\right]$$

Therefore, for maximum volume, height of cylinder is equal to diameter of its base.

Q.13. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Ans.

Let r and h be the radius and height of right circular cylinder inscribed in a given cone of radius R and height H. If S be the curved surface area of cylinder then

$S = 2\pi rh$

$$\Rightarrow S = 2\pi r.\frac{(R - r)}{R}.H \Rightarrow S = \frac{2\pi H}{R}(\text{rR} - r^2)$$

$$\left[\begin{array}{l} \because \Delta AOC \sim \Delta FEC \\ \Rightarrow \quad \frac{OC}{EC} = \frac{AO}{FE} \\ \Rightarrow \quad \frac{R}{R-r} = \frac{H}{h} \\ \Rightarrow \quad h = \frac{(R-r).H}{R} \end{array}\right]$$

Differentiating both sides

with respect to *r*, we get

$$\frac{dS}{dr} = \frac{2\pi H}{R}(R - 2r)$$

For maxima and minima

$$\Rightarrow \frac{dS}{dr} = 0 \quad \Rightarrow \quad \frac{2\pi H}{R}(R - 2r) = 0$$
$$R - 2r = 0 \quad \Rightarrow \quad r = \frac{R}{2}$$

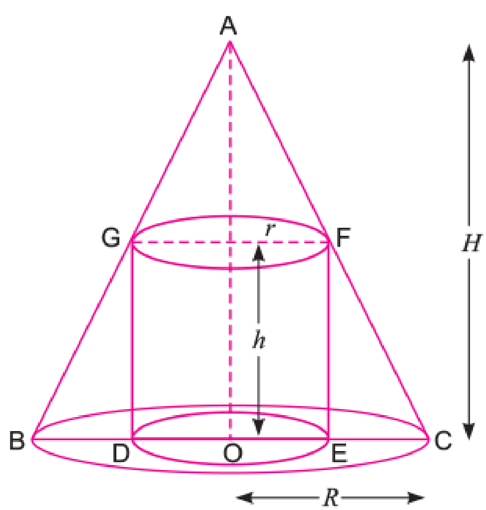
Now,

$$\Rightarrow \frac{d^2S}{dr^2} = \frac{2\pi H}{R}(0 - 2)$$

$$\Rightarrow \left[\frac{d^2S}{dr^2}\right]_{r=R/2} = -\frac{4\pi H}{R} = -\text{ve}$$

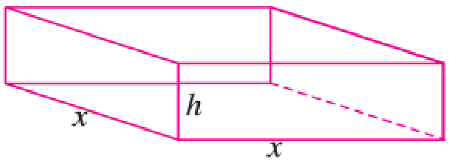
Hence, for $r = \frac{R}{2}$, S is maximum.

i.e., radius of cylinder is half of that of cone.



Q.14. An open box with a square base is to be made out of a given quantity of cardboard of area c^2square units. Show that the maximum volume of the box is $\frac{C^3}{6\sqrt{3}}$ cubic units.

Ans.



Let the length, breadth and height of open box with square be x, x and h unit respectively.

If V be the volume of box then $V = x.x.\ h \Rightarrow V = x^2h$ ....(i)

Also $c^2 = x^2 + 4xh \Rightarrow h = \frac{c^2 - x^2}{4x}$

Putting it in (i), we get

$V = \frac{x^2(c^2 - x^2)}{4x} \Rightarrow V = \frac{c^2x}{4} - \frac{x^3}{4}$

Differentiating with respect to x, we get

$\frac{dV}{dx} = \frac{c^2}{4} - \frac{3x^2}{4}$

Now for maxima or minima $\frac{dV}{dx} = 0$

$\Rightarrow \frac{c^2}{4} - \frac{3x^2}{4} = 0$

$\Rightarrow \frac{3x^2}{4} = \frac{c^2}{4}$

$\Rightarrow x^2 = \frac{c^2}{3}$

$\Rightarrow x = \frac{c}{\sqrt{3}}$

Now, $\frac{d^2V}{dx^2} = -\frac{6x}{4} = -\frac{3x}{2}$

$$\therefore \left[\frac{d^2V}{dx^2}\right]_{x=c/\sqrt{3}} = -\frac{3c}{2\sqrt{3}} = -\text{ve}$$

Hence, for $x = \frac{c}{\sqrt{3}}$ volume of box is maximum.

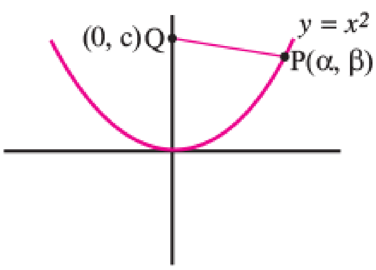
$$\therefore h = \frac{c^2 - x^2}{4x} = \frac{c^2 - \frac{c^2}{3}}{4\frac{c}{\sqrt{3}}} = \frac{2c^2}{3} \times \frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

Therefore maximum volume = $x^2 \,.\, h$

$$= \frac{c^2}{3} \cdot \frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}} \text{ cubic units}$$

Q.15. Find the shortest distance of the point (0, *c*) from the parabola $y = x^2$, where $1 \leq c \leq 5$.

Ans.



Let $P(\alpha, \beta)$ be required point on parabola $y = x^2$ such that the distance of P to given point $Q(0, c)$ is shortest.

Let $PQ = D$

$$\therefore \quad D = \sqrt{(\alpha - 0)^2 + (\beta - c)^2}$$

$$\Rightarrow \quad D^2 = \alpha^2 + (\beta - c)^2$$

$$\Rightarrow D^2 = \alpha^2 + (\alpha^2 - c)^2 \quad [\because (\alpha, \beta) \text{ lie on } y = x^2 \Rightarrow \beta = \alpha^2]$$

Now, $\frac{d(D^2)}{d\alpha} = 2\alpha + 2(\alpha^2 - c).2\alpha$
$= 2\alpha(1 + 2\alpha^2 - 2c) = 2\alpha + 4\alpha^3 - 4\alpha c$

For extremum value of D or D^2

$\frac{d(D)^2}{d\alpha} = 0$

$\Rightarrow 2\alpha(1 + 2\alpha^2 - 2c) = 0$

$\Rightarrow$ $\alpha = 0,$ or $1 + 2\alpha^2 - 2c = 0$
$\Rightarrow \alpha = 0,$ or $\alpha = \pm\sqrt{\frac{2c-1}{2}}$

Again $\frac{d^2(D^2)}{d\alpha^2} = 2 + 12\alpha^2 - 4c$

$\Rightarrow \left.\frac{d^2(D^2)}{d\alpha^2}\right] = 2 - 4c = -ve$ $[\because 1 \le c \le 5]$

$\left[\frac{d^2(D^2)}{d\alpha^2}\right]_{\alpha=\pm\sqrt{\frac{2c-1}{2}}} = 2 + 12\left(\frac{2c-1}{2}\right) - 4c$

$= 2 + 12c - 6 - 4c = 8c - 4 > 0$ $[\because 1 \le c \le 5]$

i.e., for $\alpha = \pm\sqrt{\frac{2c-1}{2}} D^2$ *i.e.*, D is minimum (shortest)

Hence required points are $\left(\pm\sqrt{\frac{2c-1}{2}}, \frac{2c-1}{2}\right)$.

Q.16. Show that the volume of the greatest cylinder that can be inscribed in a cone of height '*h*' and semi-vertical angle 'α' is.

Ans.

Let a cylinder of base radius r and height h_1 is included in a cone of height h and semi-vertical angle α.

Then $AB = r$, $OA = (h - h_1)$,

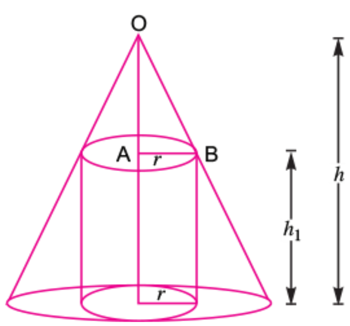
In right angle triangle OAB,

$\frac{AB}{OA} = \tan\alpha \quad \Rightarrow \quad \frac{r}{h-h_1}\tan\alpha$

or $r = (h - h_1) \tan \alpha$

$\therefore V = \pi \left[(h - h_1) \tan \alpha\right]^2 . h_1$ $\quad$ ($\because$ Volume of cylinder $= \pi r^2 h$)

$V = \pi \tan^2 \alpha . h_1 (h - h_1)^2$ $\quad$ ...(i)



Differentiating with respect to h_1, we get

$$\frac{dV}{dh_1} = \pi\tan^2 \alpha[h_1.2(h - h_1)(-1) + (h - h_1)^2 \times 1]$$

$$= \pi \tan^2 \alpha (h - h_1) [-2h_1 + h - h_1]$$

$$= \pi \tan^2 \alpha (h - h_1) (h - 3h_1)$$

For maximum volume $V, \frac{dV}{dh_1} = 0$

$\Rightarrow h - h_1 = 0 \text{ or } h - 3h_1 = 0$

$\Rightarrow \quad h = h_1 \text{ or } h_1 = \frac{1}{3}h$

$\Rightarrow h_1 = \frac{1}{3}h$ $\quad$ ($\because h = h_1$ is not possible)

Again differentiating with respect to h_1, we get

$$\frac{d^2V}{dh_1^2} = \pi\tan^2\alpha[h - h_1)(-3) + (h - 3h_1)(-1)]$$

At $h_1 = \frac{1}{3}h$,

$$\frac{d^2V}{dh_1^2} = \pi\tan^2\alpha\left[\left(h - \frac{1}{3}h\right)(-3) + 0\right]$$
$$= -2\pi h\tan^2\alpha < 0$$

$\therefore$ Volume is maximum for $h_1 = \frac{1}{3}h$

$$V_{\max} = \pi\tan^2\alpha.\left(\frac{1}{3}h\right)\left(h - \frac{1}{3}h\right)^2 \qquad \text{[Using } (i)\text{]}$$

$$= \frac{4}{27}\pi h^3\tan^2\alpha$$

Q.17. The sum of the perimeter of a circle and a square is *k*, where *k* is some constant. Prove that the sum of their areas is least when the side of the square is double the radius of the circle.

Ans.

Let side of square be a units and radius of circle be r units.

It is given that $4a + 2\pi r = k$, where k is a constant

$$\Rightarrow r = \frac{k-4a}{2\pi}$$

Sum of areas, $A = a^2 + \pi r^2$

$$\Rightarrow A = a^2 + \pi\left[\frac{k-4a}{2\pi}\right]^2 = a^2 + \frac{1}{4\pi}(k-4a)^2$$

Differentiating with respect to a, we get

$$\frac{dA}{da} = 2a + \frac{1}{4\pi}.2(k - 4a).(-4) \qquad \ldots(i)$$
$$= 2a - \frac{2(k-4a)}{\pi}$$

For minimum area, $\frac{dA}{da} = 0$

$\Rightarrow \quad 2a - \frac{2(k-4a)}{\pi} = 0$

$\Rightarrow \quad 2a = \frac{2(k-4a)}{\pi}$

$\Rightarrow \; 2a = \frac{2(2\pi r)}{\pi}$

[As $k = 4a + 2\pi r$ given]

$\Rightarrow \; a = 2r$

Now, again differentiating equation (i) with respect to a

$$\frac{d^2A}{da^2} = 2 - \frac{2}{\pi}(-4) = 2 + \frac{8}{\pi}$$

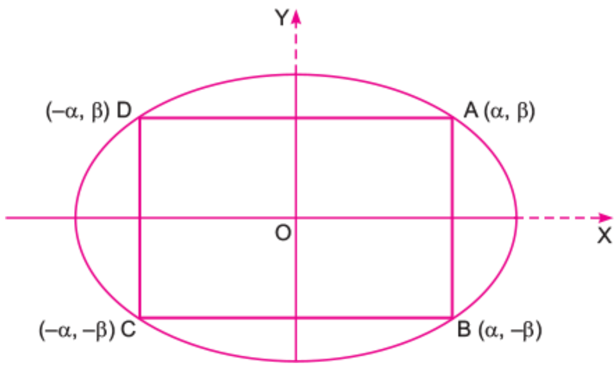
at $a = 2\pi$, $\frac{d^2A}{da^2} = 2 + \frac{8}{\pi} > 0$

$\therefore$ For $ax = 2r$, sum of areas is least.

Hence, sum of areas is least when side of the square is double the radius of the circle.

Q.18. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ans.



Let $ABCD$ be rectangle having area A inscribed in an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(i)$$

Let the coordinate of A be (α, β)

$\therefore$ Coordinate of $B \equiv (\alpha, -\beta)$, $C \equiv (-\alpha, -\beta)$, $D \equiv (-\alpha, \beta)$

Now A = Length × Breadth = $2\alpha \times 2\beta$

$A = 4\alpha\beta$

$$\Rightarrow A = 4\alpha.\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)}$$

$$\left[\begin{array}{l}\because\ (\alpha, \beta) \text{ lies on ellipse } (i) \\ \therefore\ \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \ i.e., \ \beta=\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)}\end{array}\right]$$

$$\Rightarrow A^2 = 16\alpha^2\left\{b^2\left(1-\frac{\alpha^2}{a^2}\right)\right\}$$

$$\Rightarrow \quad A^2 = \frac{16b^2}{a^2}(a^2\alpha^2-\alpha^4)$$

$$\Rightarrow \frac{d(A^2)}{d\alpha} = \frac{16b^2}{a^2}(2a^2\alpha-4\alpha^3)$$

For maximum or minimum value

$$\frac{d(A^2)}{d\alpha} = 0$$

$$\Rightarrow 2a^2\alpha - 4\alpha^3 = 0 \qquad \Rightarrow \quad 2\alpha(a^2 - 2\alpha^2) = 0$$

$$\Rightarrow \alpha = 0,\ \alpha = \frac{a}{\sqrt{2}}$$

Again $\frac{d^2(A^2)}{d\alpha^2} = \frac{16b^2}{a^2}(2a^2 - 12\alpha^2)$

$$\Rightarrow \left.\frac{d^2(A^2)}{d\alpha^2}\right]_{\alpha=\frac{a}{\sqrt{2}}} = \frac{16b^2}{a^2}\left(2a^2 - 12 \times \frac{a^2}{2}\right) < 0$$

$\Rightarrow$ For $\alpha = \frac{a}{\sqrt{2}}, A^2 i.e., A$ is maximum.

i.e., for greatest area A

$\alpha = \frac{a}{\sqrt{2}}$ and $\beta = \frac{b}{\sqrt{2}}$ (using (i))

$\therefore$ Greatest area $= 4\alpha\,.\,\beta = 4\frac{a}{\sqrt{2}} \times \frac{b}{\sqrt{2}} = 2ab$

Q.19. Tangent to the circle $x^2 + y^2 = 4$ at any point on it in the first quadrant makes intercepts *OA*and *OB* on *x* and *y* axes respectively, *O* being the centre of the circle. Find the minimum value of (*OA* + *OB*).

Ans.

Let AB be the tangent in the first quadrant to the circle $x^2 + y^2 = 4$ which make intercepts OA and OB on x and y axis respectively. Let $S = OA + OB$.

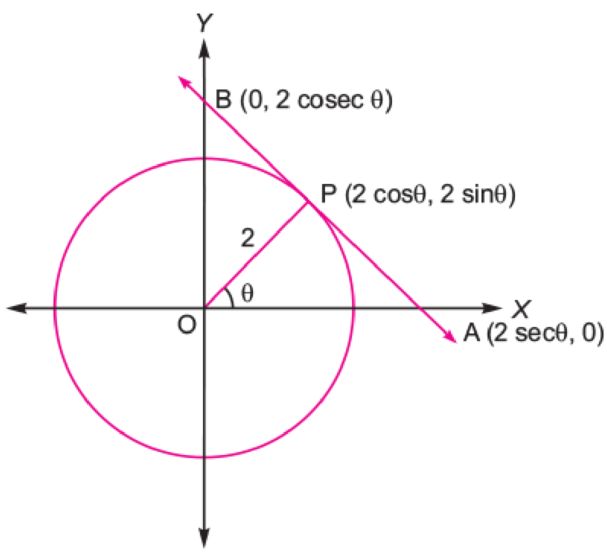
$S = OA + OB$ ... (*i*)

Let q be the angle made by OP with positive direction of x-axis.

$\therefore$Coordinates of $P = (2 \cos \theta, 2 \sin \theta)$

Coordinates of $A = (2 \sec \theta, 0)$

Coordinates of $B = (0, 2 \operatorname{cosec} \theta)$



$$\Rightarrow \frac{dS}{d\theta} = 2\left\{\sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta\right\}$$

For extremum value of V

$$\Rightarrow \frac{dS}{d\theta} = 0$$

$$\Rightarrow 2\{\sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta\} = 0$$

$$\Rightarrow \sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta = 0$$

$$\Rightarrow \frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta} = \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \frac{\sin\theta}{\cos^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

$\Rightarrow \sin^3\theta = \cos^3\theta \qquad \Rightarrow \sin\theta = \cos\theta$

$\Rightarrow \theta = \frac{\pi}{4}$ $\left[\because \theta \text{ lies in first quadrant } \Rightarrow 0 \le \theta \le \frac{\pi}{4}\right]$

Now, $\frac{d^2S}{d\theta^2} = 2\left\{(\sec^3\theta + \tan^2\theta\sec\theta) + (\text{cosec}^3\theta + \text{cosec}\,\theta\cot^2\theta)\right\}$

$\Rightarrow \left.\frac{d^2S}{d\theta^2}\right]_{\theta=\frac{\pi}{4}} = +\text{ve} \qquad \Rightarrow S$ is minimum when

$\therefore$ Minimum value of $S = OA + OB$ is $2\sec\frac{\pi}{4} + 2\text{cosec}\frac{\pi}{4} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$ units.

Q.20. Find the absolute maximum and absolute minimum values of the function *f* given by $f(x) = \sin^2 x - \cos x$, $x \in [0, \pi]$.

Ans.

Here, $f(x) = \sin^2 x - \cos x$

$f'(x) = 2\sin x.\cos x + \sin x$ $\Rightarrow f'(x) = \sin x(2\cos x + 1)$

For critical point: $f'(x) = 0$

$$\Rightarrow \quad \sin x(2\cos x + 1) = 0$$
$$\Rightarrow \quad \sin x = 0 \quad \text{or} \quad \cos x = -\frac{1}{2}$$
$$\Rightarrow \quad x = 0 \quad \text{or} \quad \cos x = \cos\frac{2\pi}{3}$$
$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 2n\pi \pm \frac{2\pi}{3}$$

where $n = 0, \pm 1, \pm 2$ ...

$\Rightarrow x = 0$ or $x = \frac{2\pi}{2}$ other values does not belong to $[0, \pi]$

For absolute maximum or minimum values:

$f(0) = \sin^2 0 - \cos 0 = 0 - 1 = -1$

$$f\left(\frac{2\pi}{3}\right) = \sin^2\frac{2\pi}{3} - \cos\frac{2\pi}{3}$$
$$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$f(\pi) = \sin^2 \pi - \cos \pi = 0 - (-1) = 1$

Hence, absolute maximum value $= \frac{5}{4}$ and absolute minimum value $= -1$.

Q.21. If the function $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1$, where $m > 0$ attains its maximum and minimum at p and q respectively such that $p^2 = q$, then find the value of m.

Ans.

Given, $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1$

$\Rightarrow f'(x) = 6x^2 - 18mx + 12m^2$

For extremum value of $f(x)$, $f'(x) = 0$

$\Rightarrow 6x^2 - 18mx + 12m^2 = 0 \qquad \Rightarrow x^2 - 3mx + 2m^2 = 0$

$\Rightarrow x^2 - 2mx - mx + 2m^2 = 0 \qquad \Rightarrow x(x - 2m) - m(x - 2m) = 0$

$\Rightarrow (x - m)(x - 2m) = 0 \qquad \Rightarrow x = m$ or $x = 2m$

Now, $f''(x) = 12x - 18m$

$\Rightarrow f''(x)$ at $[x = m] = f''(m) = 12m - 18m = -6m < 0$

And, $f''(x)$ at $[x = 2m] = f''(2m) = 24m - 18m = 6m > 0$

Hence, $f(x)$ attains maximum and minimum value at m and $2m$ respectively.

$\Rightarrow m = p$ and $2m = q$

But, $p^2 = q$ [Given]

$\therefore m^2 = 2m \qquad \Rightarrow \qquad m^2 - 2m = 0$

$\Rightarrow m(m - 2) \qquad \Rightarrow \qquad m = 0$ or $m = 2$

$\Rightarrow m = 2$ as $m > 0$

Q.22. The sum of the surface areas of a cuboid with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes.

Ans.

Let r be the radius of sphere and S, V be the sum of surface area and volume of cuboid and sphere.

Now $V = \left(x.2x.\frac{x}{3}\right) + \frac{4}{3}\pi r^3$

$\Rightarrow V = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$

$\Rightarrow V = \frac{2}{3}(x^3 + 2\pi r^3)$

$\Rightarrow V = \frac{2}{3}\left\{\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{3}{2}} + 2\pi r^3\right\}$

$\Rightarrow \frac{\mathrm{dV}}{\mathrm{dr}} = \frac{2}{3}\left\{\frac{3}{2}\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}}.\frac{1}{6}.(-8\pi r) + 6\pi r^2\right\}$

$$\left[\begin{array}{l} \because \quad S = 2\left[x.2x + x.\frac{x}{3} + \frac{x}{3}.2x\right] + 4\pi r^2 \\ \Rightarrow \quad S = \frac{18x^2}{3} + 4\pi r^2 = 6x^2 + 4\pi r^2 \\ \Rightarrow \quad x^2 = \frac{S - 4\pi r^2}{6} \quad \Rightarrow \quad x^3 = \left(\frac{S - 4\pi r^2}{6}\right)^{3/2} \end{array}\right.$$

For maximum or minimum value

$\frac{\mathrm{dV}}{\mathrm{dr}} = 0$

$\Rightarrow \quad \frac{2}{3}\left\{-2\pi r\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} + 6\pi r^2\right\} = 0$

$\Rightarrow \quad \left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} = \frac{6\pi r^2}{2\pi r}$

$\Rightarrow \quad \left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} = 3r$

$\Rightarrow \quad r = \frac{1}{3}.\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}}$

Obviously, $\left.\frac{d^2V}{dr^2}\right]_{r=\frac{1}{3}\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}} = +\text{ve}$

$\therefore \quad V$ is minimum when $r = \frac{1}{3}\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}$

$\Rightarrow \quad 3r = \left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}$

$\Rightarrow \quad 9r^2 = \left(\frac{S-4\pi r^2}{6}\right) \quad \Rightarrow \quad 54r^2 = S - 4\pi r^2$

$\Rightarrow \; 54r^2 = 6x^2 + 4\pi r^2 - 4\pi r^2 \; [\because S = 6x^2 + 4\pi r^2]$

$\Rightarrow \; x^2 = 9r^2 \qquad \Rightarrow \qquad x = 3r$

i.e., x is equal to three times the radius of sphere.

Now Minimum value of V (sum of volume) $= \frac{2}{3}\left\{x^3 + 2\pi\left(\frac{x}{3}\right)^3\right\}$

$= \frac{2}{3}\left\{x^3 + \frac{2\pi}{27}x^3\right\} = \frac{2}{81}x^3(27 + 2\pi)$ cubic unit.

Q.23. Find the maximum and minimum values of $f(x) = \sec x + \log \cos^2 x$, $0 < x < 2\pi$.

Ans.

We have

$f(x) = \sec x + \log \cos^2 x$

$f'(x) = \sec x . \tan x + \frac{1}{\cos^2 x} . 2\cos x(-\sin x) = \sec x . \tan x - 2\tan x = \tan x(\sec x - 2)$

For critical point

$f'(x) = 0$

$\Rightarrow \tan x(\sec x - 0) = 0 \qquad \Rightarrow \tan x = 0$ or $\sec x - 2 = 0$

$\Rightarrow \quad x = n\pi \quad \text{or} \quad \sec\ x = 2$

$\Rightarrow \quad n\pi \quad \text{or} \quad \cos\ x = \frac{1}{2}$

$\Rightarrow \quad x = n\pi \quad \text{or} \quad \cos\ x = \cos\frac{\pi}{3}$

$\Rightarrow \quad x = n\pi \quad \text{or} \quad x = 2n\pi \pm \frac{\pi}{3},\ n = 0,\ \pm\ 1, \pm\ 2.......$

Thus possible value of x in interval $0 < x < 2\pi$ are

$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Now, $f\left(\frac{\pi}{3}\right) = \sec\frac{\pi}{3} + \log\cos^2\frac{\pi}{3} = 2 + \log\left(\frac{1}{2}\right)^2$

$= 2 + 2(\log 1 - \log 2) = 2 - 2\log 2 = 2(1 - \log 2) \qquad [\because \log 1 = 0]$

$f(\pi) = \sec\pi + \log\cos^2\pi = -1 + \log(-1)^2 = -1$

$f\left(\frac{5\pi}{3}\right) = \sec\frac{5\pi}{3} + 2\log\ \cos\ \frac{5\pi}{3}$

$= \sec\ \left(2\pi - \frac{\pi}{3}\right) + 2\log\ \cos\left(2\pi - \frac{\pi}{3}\right)$

$= \sec\ \frac{\pi}{3} + 2\log\ \cos\frac{\pi}{3} = 2 + 2\log\frac{1}{2}$

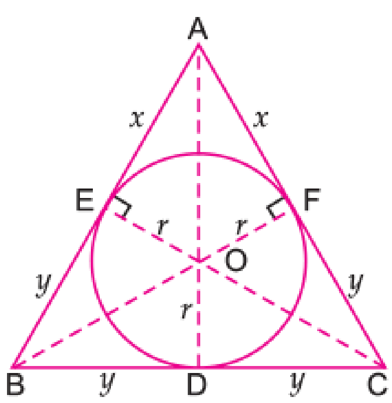
$= 2 + 2(\log 1 - \log 2) = 2 - 2\log 2 = 2(1 - \log 2)$

Hence,Maximum value of $f(x) = 2(1 - \log 2)$

Minimum value of $f(x) = -1$

Q.24. Prove that the least perimeter of an isosceles triangle in which a circle of radius *r* can be inscribed is $6\sqrt{3r}$.

Ans.



Let ΔABC be isosceles triangle having $AB = AC$ in which a circle with centre O and radius r is inscribed touching sides AB, BC and AC at E, D and F respectively.

Let $AE = AF = x$, $BE = BD = y$

Obviously, $CF = CD = y$

Let P be the perimeter of ΔABC.

$$\therefore \quad P = 2x + 4y \quad \Rightarrow P = \frac{4yr^2}{y^2 - r^2} + 4y$$

Differentiating w.r.t. y, we get

$$\Rightarrow \quad \frac{dP}{dy} = \frac{(y^2 - r^2).4r^2 - 4yr^2(2y - 0)}{(y^2 - r^2)^2} + 4$$

$$\Rightarrow \quad \frac{dP}{dy} = \frac{4y^2r^2 - 4r^4 - 8y^2r^2}{(y^2 - r^2)^2} + 4$$

$$\Rightarrow \quad \frac{dP}{dy} = \frac{-4r^2(r^2 + y^2)}{(y^2 - r^2)^2} + 4$$

For critical point $\frac{dP}{dy} = 0$

$$\Rightarrow \quad \frac{-4r^2(r^2 + y^2)}{(y^2 - r^2)^2} + 4 = 0$$

$$\Rightarrow \quad -4r^2(r^2 + y^2) + 4(y^2 - r^2)^2 = 0$$

$$\Rightarrow \quad -r^4 - r^2y^2 + y^4 + r^4 - 2y^2r^2 = 0$$

$$\Rightarrow \quad y^4 - 3r^2y^2 = 0$$

$$\Rightarrow \quad y^2[y^2 - 3r^2] = 0$$

$$\Rightarrow \quad y = \sqrt{3}r \quad [\because y \neq 0]$$

Also $\left.\frac{d^2P}{dr^2}\right]_{\sqrt{3}r} = +\text{ve}$

$$ar\,(\Delta ABC) = ar\,(\Delta BOC) + ar\,(\Delta AOC) + ar\,(\Delta AOB)$$
$$\Rightarrow \quad \frac{1}{2} AD.\,BC = \frac{1}{2}.\,BC.\,OD + \frac{1}{2}.\,AC\,.\,OF + \frac{1}{2}.\,AB.\,OE$$
$$\Rightarrow \quad 2y.\,(r + \sqrt{r^2 + x^2}) = 2y.\,r + (x+y).\,r + (x+y)\,.\,r$$
$$\Rightarrow 2y.\,(r + \sqrt{r^2 + x^2}) = 2yr + 2(x+y)\,.\,r$$
$$\Rightarrow \quad yr + y\sqrt{r^2 + x^2} = yr + xr + yr$$
$$\Rightarrow \quad y\sqrt{r^2 + x^2} = xr + yr$$
$$\Rightarrow \quad y^2(r^2 + x^2) = x^2r^2 + y^2r^2 + 2xyr^2$$
$$\Rightarrow \quad y^2r^2 + x^2y^2 = x^2r^2 + y^2r^2 + 2xyr^2$$
$$\Rightarrow \quad x^2y^2 = x^2r^2 + 2xyr^2$$
$$\Rightarrow \quad xy^2 = xr^2 + 2yr^2$$

$\Rightarrow$ when $y = \sqrt{3}r$, the value of P is minimum.

$\therefore$ Least perimeter $= 4y + \frac{4r^2y}{y^2 - r^2} = 4\sqrt{3}r + \frac{4r^2.\sqrt{3}r}{3r^2 - r^2}$

$= 4\sqrt{3}r + \frac{4\sqrt{3}r^3}{2r^2} = 6\sqrt{3}r$ units

Q.25. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which pass through the point $\left(\frac{4}{3}, 0\right)$

Ans.

Let the point of contact be (x_0, y_0)

Now given curve is $3x^2 - y^2 = 8$

Differentiating w.r.t. x we get, $6x - 2y.\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{6x}{2y} = \frac{3x}{y} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right]_{(x_0, y_0)} = \frac{3x_0}{y_0}$

Now, equation of required tangent is

$(y - y_0) = \frac{3x_0}{y_0}(x - x_0) \qquad \ldots(i)$

$\because$ (i) passes through $\left(\frac{4}{3}, 0\right)$

$\therefore (0 - y_0) = \frac{3x_0}{y_0}\left(\frac{4}{3} - x_0\right) \quad \Rightarrow -y_0^2 = 4x_0 - 3x_0^2$

(ii)

Also, $\because (x_0, y_0)$ lie on given curve $3x^2 - y^2 = 8$

$\Rightarrow \quad 3x_0^2 - y_0^2 = 8 \quad \Rightarrow \quad y_0^2 = 3x_0^2 - 8$

Putting y_0^2 in (ii) we get

$-(3x_0^2 - 8) = 4x_0 - 3x_0^2 \quad \Rightarrow 4x_0 = 8 \quad \Rightarrow \quad x_0 = 2$

$\therefore \quad y_0 = \sqrt{3 \times 2^2 - 8} = \sqrt{4} = \pm 2$

Therefore, equations of required tangents are

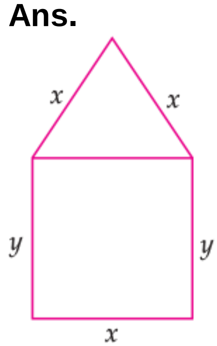
$(y - 2) = \frac{3 \times 2}{2}(x - 2)$ and $(y + 2) = \frac{3 \times 2}{-2}(x - 2)$

$\Rightarrow y - 2 = 3x - 6$ and $y + 2 = -3x + 6$

$\Rightarrow 3x - y - 4 = 0$ and $3x + y - 4 = 0$

Q.26. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window.

Ans.



Let x and y be the dimensions of rectangular part of window and x be side of equilateral part.

If A be the total area of window, then $A = x.y + \frac{\sqrt{3}}{4}x^2$ ...(*i*)

Also, $x + 2y + 2x = 12$ $\Rightarrow 3x + 2y = 12$

$\Rightarrow y = \frac{12-3x}{2}$

$\therefore A = x.\frac{(12-3x)}{2} + \frac{\sqrt{3}}{4}x^2$ [From (*i*)]

$\Rightarrow A = 6x - \frac{3x^2}{2} + \frac{\sqrt{3}}{4}x^2$

$\Rightarrow A' = 6 - 3x + \frac{\sqrt{3}}{2}x$ [Differentiating with respect to x]

Now, for maxima or minima

$A' = 0 \Rightarrow 6 - 3x + \frac{\sqrt{3}}{2}x = 0 \Rightarrow x = \frac{12}{6-\sqrt{3}}$

Again $A'' = -3 + \frac{\sqrt{3}}{2} < 0$ (for any value of x)

$\Rightarrow A''\big]_{x=\frac{12}{6-\sqrt{3}}} < 0$

i.e., is maximum if $x = \frac{12}{6-\sqrt{3}}$ and $y = \frac{12-3\left(\frac{12}{6-\sqrt{3}}\right)}{2}$.

i.e., For largest area of window, dimensions of rectangle are

$x = \frac{12}{6-\sqrt{3}}$ and $y = \frac{18-6\sqrt{3}}{6-\sqrt{3}}$.

Q.27. Show that the normal at any point to the curve $x = acos\theta + a\theta sin\theta, y = asin\theta - a\theta cos\theta$ is at a constant distance from the origin.

Ans.

Given $x = a\ \cos\theta + a\ \theta \sin\theta$

$y = a \sin\theta - a\ \theta\ \cos\theta$

$\therefore \quad \frac{dx}{d\theta} = -\ a\ \sin\theta + a\ (\theta\ \cos\theta +\ \sin\theta)$

$= -\ a \sin\theta + a\ \theta\ \cos\theta + a \sin\theta = a\ \theta\ \cos\theta$ and

$\frac{dy}{d\theta} = a\ \cos\theta - a(-\ \theta \sin\theta\ +\ \cos\theta)$

$= a \cos\theta + a\ \theta \sin\theta - a \cos\theta = a\ \theta \sin\theta$

$\Rightarrow \ \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\ \theta\ \sin\theta}{a\ \theta\ \cos\theta} = \tan\theta$

$\therefore$ Slope of tangent at $\theta = \tan\theta$ $\qquad \Rightarrow$ Slope of normal at $\theta = -\frac{1}{\tan\theta} = -\cot\theta$

Hence equation of normal at θ is

$\frac{y - (a \sin\theta - a\ \theta\ \cos\theta)}{x - (a\ \cos\theta + a\ \theta\ \sin\theta)} = -\ \cot\theta$

$\Rightarrow y - a\ \sin\theta + a\ \theta\ \cos\theta + x \cot\theta - \cot\theta\ (a \cos\theta + a\ \theta \sin\theta) = 0$

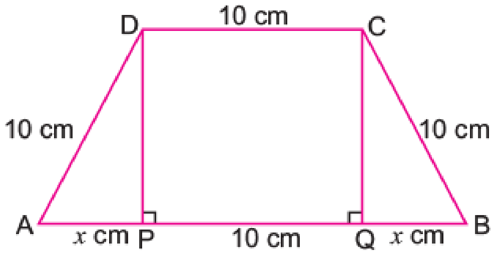
$\Rightarrow y - a\ \sin\theta + a\ \theta\ \cos\theta + x \frac{\cos\theta}{\sin\theta} - a \frac{\cos^2\theta}{\sin\theta} - a\theta \cos\theta = 0$

$\Rightarrow x \cos\theta + y \sin\theta - a = 0$

Distance from origin $(0, 0)$ to (i) $= \left| \frac{0.\cos\theta + 0.\sin\theta - a}{\sqrt{\cos^2\theta + \sin^2\theta}} \right| = a$

Q.28. If length of three sides of a trapezium other than base are equal to 10 cm, then find the area of the trapezium when it is maximum.

Ans.



The required trapezium is as given in figure. Draw perpendiculars *DP* and *CQ* on *AB*. Let $AP = x$ cm. Note that $\Delta APD \cong \Delta BQC$. Therefore, $QB = x$ cm. Also, by Pythagoras theorem $DP = QC = \sqrt{100 - x^2}$. Let *A* be the area of the trapezium.

Then, $A \equiv A(x) = \frac{1}{2}$ (sum of parallel sides) × (height)

$$= \frac{1}{2}(2x + 10 + 10)(\sqrt{100 - x^2}) = (x + 10)\sqrt{100 - x^2}$$

or $A'(x) = (x + 10)\frac{(-2x)}{2\sqrt{100 - x^2}} + (\sqrt{100 - x^2})$

$$= \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}$$

Now $A'(x) = 0$ gives $2x^2 + 10x - 100 = 0$, *i.e.*, $x = 5$ and $x = -10$

Since x represents distance, it cannot be negative.

So, $x = 5$.

$$A''(x) = \frac{\sqrt{100-x^2}(-4x-10)-(-2x^2-10x+100)\frac{(-2x)}{2\sqrt{100-x^2}}}{100-x^2}$$

$$= \frac{2x^3-300x-1000}{(100-x^2)^{\frac{3}{2}}} \quad \text{(on simplification)}$$

or $A''(5) = \frac{2(5)^3-300(5)-1000}{(100-(5)^2)^{\frac{3}{2}}} = \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$

Thus, area of trapezium is maximum at $x = 5$ and the maximum area is given by

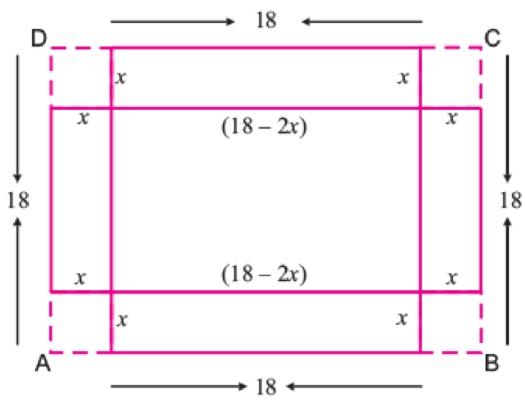
$$A(5) = (5 + 10)\sqrt{100 - (5)^2} = 15\sqrt{75} = 75\sqrt{3}\text{cm}^2$$

Long Answer Questions-II (OIQ)

[6 Marks]

Q.1. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. Find the maximum volume of the box.

Ans.



Let x be the side of the square which is to be cut off from each corner.

Then dimensions of the box are $18 - 2x$, $18 - 2x$ and x.

Let V be the volume of the box then

$$V = x(18 - 2x)(18 - 2x) = x(18 - 2x)^2$$

$$\Rightarrow V' = x\frac{d}{dx}(18 - 2x)^2 + (18 - 2x)^2\frac{d}{dx}x$$

$$= 2x(18 - 2x)(-2) + (18 - 2x)^2$$

$$= (18 - 2x)[-4x + 18 - 2x]$$

$$\Rightarrow V' = (18 - 2x)(18 - 6x)$$

Also, $V'' = (18-2x)\frac{d}{dx}(18-6x) + (18-6x)\frac{d}{dx}(18-2x)$

$\Rightarrow V' = (18-2x)(-6) + (18-6x)(-2)$

For maximum or minimum value $V' = 0$

$\Rightarrow (18-2x)\ (18-6x) = 0 \Rightarrow x = 9 \quad \text{or} \quad x = 3$

Neglecting $x = 9$ $[\because$ For $x = 9$, length $= 18 - 2x = 18 - 2(9) = 0]$

Therefore, $x = 3$ is to be taken.

$V'(3) = (18-6)(-6) + (18-18)(-2) = 72 < 0$

Thus, volume is maximum when $x = 3$

$\therefore$ Length $= 18 - 2x = 18 - 6 = 12$ cm; Breadth $= 18 - 2x = 18 - 6 = 12$ cm; Height $= x = 3$ cm

Maximum volume of the box $= 12 \times 12 \times 3 = 432\ \text{cm}^3$.

Q.2. A wire of length 28 cm is to be cut into two pieces. One of the two pieces is to be made into a square and the other into a circle. What should be the length of two pieces so that the combined area of them is minimum?

Ans.

Let the length of one piece be x cm, then the length of other piece will be $(28-x)$ cm.

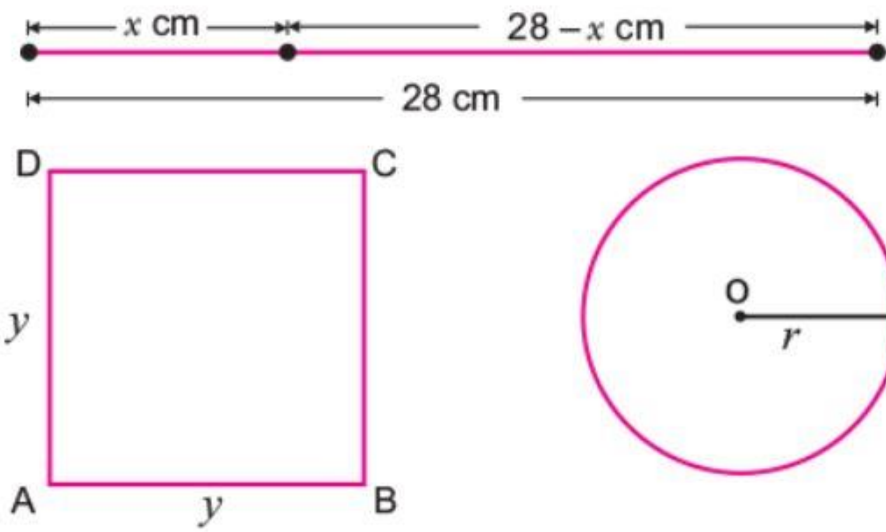
Let from the first piece we make a circle of radius r and from the second piece we make a square of side y.

Then $\begin{cases} 2\pi r = x \Rightarrow r = \frac{x}{2\pi} \\ 4y = 28 - x \Rightarrow y = \frac{(28-x)}{4} \end{cases}$ ...(i)

Let A be the combined area of the circle and square then

$A = \pi r^2 + y^2 \quad \Rightarrow A = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{28-x}{4}\right)^2$ ...(ii)

Differentiating (ii) with respect to 'x', we get



$$A' = \frac{\pi}{4\pi^2}(2x) + \frac{1}{16}(2)(28-x)(-1)$$

$$\Rightarrow A' = \frac{x}{2\pi} - \frac{1}{8}(28-x) \text{ and}$$

$$A' = \frac{1}{2\pi} - \frac{1}{8}(0-1) = \frac{1}{2\pi} + \frac{1}{8}$$

For maximum and minimum $A' = 0$

$$\Rightarrow \frac{x}{2\pi} - \frac{28}{8} + \frac{x}{8} = 0$$

$$\Rightarrow x\left(\frac{1}{2\pi} + \frac{1}{8}\right) = \frac{28}{8}$$

$$\Rightarrow \quad x\left(\frac{4+\pi}{8\pi}\right) = \frac{28}{8} \quad \Rightarrow \quad x = \frac{28\pi}{4+\pi}$$

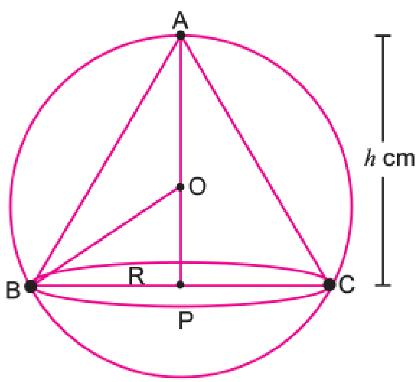
Since, $A' = +\text{ve}$ for $x = \frac{28\pi}{4+\pi}$ $\therefore A$ is min for $x = \frac{28\pi}{4+\pi}$

Thus, the required length of two pieces are

$$x = \frac{28\pi}{4+\pi}\text{ cm and } 28 - x$$
$$= 28 - \frac{28\pi}{4+\pi} = \frac{192}{4+\pi}\text{ cm}$$

Q.3. Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12 cm is 16 cm.

Ans.



Let O be the centre of a sphere of radius 12 cm and a cone ABC of radius R cm and height h cm is inscribed in the sphere.

$AP = AO + OP \quad \Rightarrow h = 12 + OP \quad \Rightarrow \quad OP = (h - 12)$

Now in right angle ΔOBP, by Pythagoras theorem, we get

$BO^2 = BP^2 + OP^2$

$(12)^2 = R^2 + (h - 12)^2 \quad \Rightarrow \quad 144 = R^2 + h^2 + 144 - 24h \quad \Rightarrow \quad R^2 = 24h - h^2$

Volume of cone,

$\Rightarrow \quad V = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi(24h - h^2)h$

$V = \frac{1}{3}\pi(24h^2 - h^3)$

$\Rightarrow \quad \frac{dV}{dh} = \frac{\pi}{3}(48h - 3h^2)$

For maximum/minimum value of V, we have

$\frac{dV}{dh} = 0 \Rightarrow \quad 48h - 3h^2 = 0$

$\Rightarrow \quad h(48 - 3h) = 0 \Rightarrow \quad$ either $h = 0$ or $h = 16$

But height of cone cannot be zero.

Therefore $h = 16$ cm.

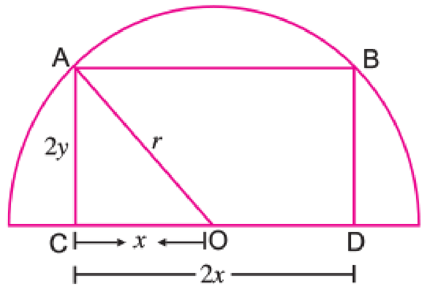
Now, $\frac{d}{dh}\left(\frac{dV}{dh}\right) = \frac{\pi}{3}(48 - 6h)$

$\Rightarrow \quad \left(\frac{d^2V}{dh^2}\right)_{h=16} = \frac{\pi}{3}(48 - 6 \times 16) = -16\pi < 0$

Hence, volume of cone is maximum when $h = 16$ cm.

Q.4. A rectangle is inscribed in a semi-circle of radius *r* with one of its sides on diameter of semi-circle. Find the dimensions of the rectangle so that its area is maximum. Find the area also.

Ans.



Let $ABCD$ be the rectangle which is inscribed in a semi-circle with centre O and radius r.

Assume that the length of rectangle is $2x$ and breadth is $2y$.

$\Rightarrow \quad CD = AB = 2x$

In right angle ΔACO,

We get $x^2 + 4y^2 = r^2$ (by Pythagoras theorem) $\quad ...(i)$

Now area of rectangle $ABCD$ is given by,

$A = \text{length} \times \text{breadth} \quad \Rightarrow \quad A = 2x \times 2y = 4xy$

$\Rightarrow A = 4x\sqrt{\frac{(r^2-x^2)}{4}} \qquad \text{[from equation } (i)\text{]}$

$\Rightarrow \; A^2 = 16x^2\frac{(r^2-x^2)}{4}$

Let $Z = A^2 = 4(r^2x^2 - x^4) \quad ...(ii)$

Then Z is maximum or minimum according as A is maximum or minimum.

Differentiating equation (ii) with respect to x, we get

$\frac{dZ}{dx} = 4[r^2.\,2x - 4x^3]$

For maximum or minimum value of Z, we have $\frac{dZ}{dx} = 0 \;\Rightarrow\; 8x(r^2 - 2x^2) = 0$

$\Rightarrow r^2 - 2x^2 = 0 \qquad (\because x \text{ cannot be zero}) \quad \Rightarrow \; x^2 = \frac{r^2}{2}$

Now, $\frac{d^2Z}{dx^2} = 4[2r^2 - 12x^2]$

$$\frac{d^2Z}{dx^2}\left(\text{at } x=\frac{r}{\sqrt{2}}\right) = 4\left[2r^2 - 12 \times \frac{r^2}{2}\right]$$

$$= -16r^2 < 0 \text{ at } x = \frac{r}{\sqrt{2}}$$

Thus area will be maximum when $x = \frac{r}{\sqrt{2}} \Rightarrow 2x = \sqrt{2}\,r,$

Putting $x = \frac{r}{\sqrt{2}}$ in equation (i), we obtain $y = \frac{r}{2\sqrt{2}}$

$\Rightarrow 2y = \frac{r}{\sqrt{2}}$

Now, $A = 2x \times 2y = \sqrt{2}r \times \frac{r}{\sqrt{2}} = r^2$.

Hence, the dimensions are $\sqrt{2}r$ and $\frac{r}{\sqrt{2}}$ and area = r^2 sq units.

Q.5. A wire of length 36 cm is cut into two pieces, one of the pieces is turned in the form of a square and other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum.

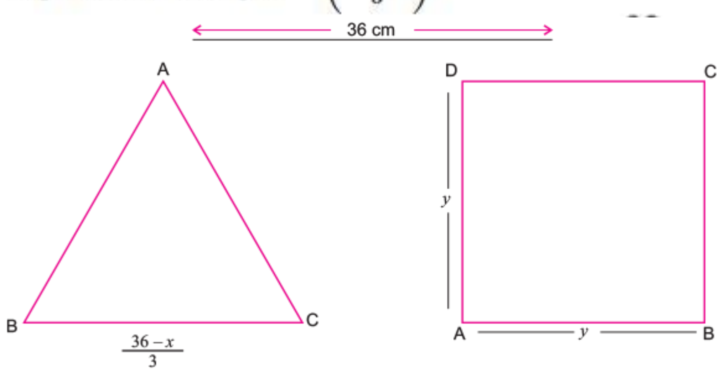
Ans.

Let the length of one piece be x, then the length of the other piece will be $36 - x$.

Let from first piece we make the square, then

$x = 4y \Rightarrow y = \frac{x}{4}$, where y is the side of the square ...(i)

From the second piece of length $(36 - x)$ we make an equilateral triangle, then side of the equilateral triangle $= \left(\frac{36-x}{3}\right)$



Now combined area of the two = $A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{36-x}{3}\right)^2$

Differentiating with respect to *x*, we have

$$\Rightarrow \quad \frac{dA}{dx} = \frac{2x}{4}.\frac{1}{4} + \frac{\sqrt{3}}{4}.2.\frac{(36-x)}{3}.\left(-\frac{1}{3}\right)$$
$$\frac{dA}{dx} = \frac{x}{8} - \frac{\sqrt{3}}{18}(36-x) \qquad ...(ii)$$

For maximum/minimum, we have $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{x}{8} - \frac{\sqrt{3}}{18}(36-x) = 0$$
$$\Rightarrow \quad \frac{x}{8} = \frac{\sqrt{3}}{18}.(36-x) \quad \Rightarrow \quad \frac{x}{8} = 2\sqrt{3} - \frac{1}{6\sqrt{3}}x$$
$$\Rightarrow \quad x\left(\frac{1}{8} + \frac{1}{6\sqrt{3}}\right) = 2\sqrt{3} \quad \Rightarrow \quad x\left(\frac{3\sqrt{3}+4}{24\sqrt{3}}\right) = 2\sqrt{3}$$
$$\Rightarrow \quad x(4+3\sqrt{3}) = 144 \quad \Rightarrow \quad x = \frac{144}{4+3\sqrt{3}}$$

Thus, length of one piece is $x = \frac{144}{4+3\sqrt{3}}$ and the length of other piece is

$$36 - \frac{144}{(4+3\sqrt{3})} = \frac{144+108\sqrt{3}-144}{(4+3\sqrt{3})}$$
$$= \frac{108\sqrt{3}}{(4+3\sqrt{3})}\text{cm}$$