Very Short Answer Questions (PYQ)

[1 Mark]

Q.1. Evaluate: $\int \frac{dx}{\sqrt{1-x^2}}$

Ans.

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ $\left[\because \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}\right]$

Q.2. Write the value of $\int \frac{dx}{x^2+16}$.

Ans.

$\int \frac{dx}{x^2+16} = \int \frac{dx}{x^2+4^2}$

$= \frac{1}{4}\tan^{-1}\frac{x}{4} + C$ $\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$

Q.3. Evaluate: $\int \frac{x^2}{1+x^3} dx$

Ans.

Let $1 + x^3 = z \quad \Rightarrow 3x^2\, dx = dz \quad \Rightarrow x^2 dx = \frac{dz}{3}$

$\therefore \int \frac{x^2}{1+x^3} dx = \frac{1}{3}\int \frac{dz}{z} = \frac{1}{3}\log|z| + C = \frac{1}{3}\log|1+x^3| + C$

Q.4. Evaluate: $\int \frac{\sec^2 x}{3+\tan x} dx$

Ans.

Let $3 + \tan x = z \Rightarrow \sec^2 x\, dx = dz$

$\therefore \int \frac{\sec^2 x\, dx}{3+\tan x} = \int \frac{dz}{z} = \log|z| + C = \log|3+\tan x| + C$

Q.5. Evaluate: $\int \sec^2 (7-x)\, dx$

Ans.

$$\int \sec^2 (7-x)\text{dx} = \frac{\tan (7-x)}{-1} + C = -\tan (7-x) + C$$

Q.6. Evaluate: $-\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}}\text{dx}$

Ans.

Let $\sqrt{x} = z \quad \Rightarrow \quad \frac{1}{2\sqrt{x}}dx = dz \quad \Rightarrow \quad \frac{dx}{\sqrt{x}} = 2dz$

$$\therefore \int \frac{\sec^2 \sqrt{x}}{\sqrt{x}}\text{dx} = 2\int \sec^2 z \text{ dz} = 2\tan z + C = 2\tan\sqrt{x} + C$$

Q.7. Evaluate: $\int \frac{\text{dx}}{x + x\log x}$

Ans.

Let $I = \int \frac{\text{dx}}{x(1+\log x)}$

Put $1 + \log x = z$, we get

$\frac{1}{x}dx = dz$

$$\therefore \quad I = \int \frac{\text{dz}}{z} = \log z + C = \log(1 + \log x) + C$$

Q.8. Evaluate: $\int \frac{\log x}{x}\text{dx}$

Ans.

Let $\log x = z \quad \Rightarrow \quad \frac{1}{x}dx = dz$

$$\int \frac{\log x}{x}\text{dx} = \int z\text{dz} = \frac{z^2}{2} + C = \frac{1}{2}(\log x)^2 + C$$

Q.9. Evaluate: $\int \frac{x + \cos 6x}{3x^2 + \sin 6x}\text{dx}$

Ans.

Let $I=\int \frac{x+\cos 6x}{3x^2+\sin 6x}\mathrm{dx}$

Let $3x^2 + \sin 6x = t$

$\Rightarrow (6x+6\cos 6x)\ dx = dt \qquad \Rightarrow \qquad (x+\cos 6x)\ dx = \frac{\mathrm{dt}}{6}$

$\therefore\ I=\int \frac{\mathrm{dt}}{6t}=\frac{1}{6}\log|t|+C=\frac{1}{6}\log\left|3x^2+\sin 6x\right|+C$

Q.10. **Evaluate:** $\int_0^1 \frac{\mathrm{dx}}{1+x^2}$

Ans.

$\int_0^1 \frac{\mathrm{dx}}{1+x^2}=[\tan^{-1}x]_0^1$

$=\tan^{-1}1-\tan^{-1}0=\frac{\pi}{4}-0=\frac{\pi}{4}$

Q.11. **If $\int_0^1 (3x^2+2x+k)\mathrm{dx}=0$, then find the value of k.**

Ans.

Given, $\int_0^1 (3x^2+2x+k)\mathrm{dx}=0 \quad \Rightarrow \quad \left[\frac{3x^3}{3}+\frac{2x^2}{2}+\mathrm{kx}\right]_0^1=0 \quad \Rightarrow$ [3 x 3 3 + 2 x 2 2 +kx] 0 1 =0

$\Rightarrow \quad [x^3+x^2+\mathrm{kx}]_0^1 \quad \Rightarrow \quad (1+1+k)-(0)=0$

$\Rightarrow \quad k=-2$

Q.12. **Evaluate:** $\int_0^{1/\sqrt{2}} \frac{\mathrm{dx}}{\sqrt{1-x^2}}$

Ans.

$I=\int_0^{1/\sqrt{2}} \frac{\mathrm{dx}}{\sqrt{1-x^2}}=[\sin^{-1}x]_0^{1/\sqrt{2}}=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)-\sin^{-1}(0)=\frac{\pi}{4}-0=\frac{\pi}{4}$

Q.13. Evaluate: $\int_2^3 \frac{1}{x} dx$

Ans.

$$\int_2^3 \frac{1}{x} dx = [\log x]_2^3 = \log 3 - \log 2 = \log \frac{3}{2}$$

Q.14. If $\int \left(\frac{x-1}{x^2}\right) e^x dx = f(x) e^x + C$, then write the value of $f(x)$.

Ans.

Given, $\int \left(\frac{x-1}{x^2}\right) e^x dx = f(x).e^x + C$

$$\Rightarrow \int \left(\frac{1}{x} - \frac{1}{x^2}\right) e^x dx = f(x).e^x + C \quad \Rightarrow \quad \frac{1}{x}.e^x + C = f(x).e^x + C$$

$$\Rightarrow \quad \frac{1}{x}.\, e^x + C = f(x).\, e^x + C$$

Equating we get $f(x) = \frac{1}{x}$

[**Note:** $\int [f(x) + f'(x)] e^x = f(x) e^x + C$]

Very Short Answer Questions (OIQ)

[1 Mark]

Q.1. Find the value of $\int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$.

Ans.

Let $f(x) = \sin^7 x$. Then $f(-x) = -\sin^7 x = -f(x)$

So, $f(x)$ is odd function.

Therefore, $\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0 \quad \Rightarrow \quad \int_{-\pi/2}^{\pi/2} \sin^7 x\,dx = 0$

Q.2. **Evaluate:** $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$

Ans.

Let $\cos x = t \quad \Rightarrow -\sin x\,dx = dt$, when $x = 0$, $t = 1$, when $x = \frac{\pi}{2}, t = 0$

$\therefore \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = \int_1^0 \frac{-dt}{1+t^2}$

$= \int_0^1 \frac{dt}{1+t^2} = [\tan^{-1} t]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

Short Answer Questions-I (PYQ)

[2 Mark]

Q.1. Evaluate: $\int \frac{(x-4)e^x}{(x-2)^3}dx$

Ans.

$$I=\int\left[\frac{(x-2)-2}{(x-2)^3}\right]e^x\,dx=\int\frac{e^x}{(x-2)^2}dx-2\int\frac{e^x}{(x-2)^3}dx$$

$$=\frac{e^x}{(x-2)^2}+2\int\frac{e^x\,dx}{(x-2)^3}-2\int\frac{e^x\,dx}{(x-2)^3}=\frac{e^x}{(x-2)^2}+C$$

Q.2. Given $\int e^x(\tan x+1)\sec x\,dx=e^xf(x)+C$

Write $f(x)$ satisfying the above.

Ans.

$$\text{Given,}\int e^x(\tan x+1)\sec x\,dx=e^xf(x)+C$$

$$\Rightarrow\int e^x(\tan x\sec x+\sec x)dx=e^xf(x)+C$$

$$\Rightarrow\int e^x(\sec x+\tan x\sec x)dx=e^xf(x)+C$$

$$\Rightarrow\quad e^x\sec x+C=e^xf(x)+C$$

$$\Rightarrow f(x)=\sec x$$

[Note: $[\int e^x[f(x)+f'(x)]dx=e^x.f(x)+C$, Here $f(x)=\sec x]$

Q.3. Evaluate: $\int(1-x)\sqrt{x}\,dx$

Ans.

$$\int (1-x)\sqrt{x}\,\mathrm{dx} = \int \sqrt{x}\,\mathrm{dx} - \int x^{1+\frac{1}{2}}\,\mathrm{dx} = \int x^{\frac{1}{2}}\,\mathrm{dx} - \int x^{\frac{3}{2}}\,\mathrm{dx}$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$$

Q.4. Evaluate $\int_0^1 \frac{\tan^{-1} x}{1+x^2}\mathrm{dx}$

Ans.

Let $t = \tan^{-1}x \quad \Rightarrow \quad dt = \frac{1}{1+x^2}dx$

Also when, $x = 0$, $t = 0$ and when $x = 1 \quad \Rightarrow t = \frac{\pi}{4}$

$\therefore \int_0^1 \frac{\tan^{-1} x}{1+x^2}\mathrm{dx} = \int_0^{\frac{\pi}{4}} t\,\mathrm{dt}$

$$= \left[\frac{t^2}{2}\right]_0^{\pi/4} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

Q.5. Evaluate: $\int_0^1 \frac{\mathrm{dx}}{\sqrt{2x+3}}\mathrm{dx}$

Ans.

Let $I = \int_0^1 \frac{\mathrm{dx}}{\sqrt{2x+3}} = \int_0^1 (2x+3)^{-1/2}\,\mathrm{dx}$

$$= \left[\frac{(2x+3)^{-1/2+1}}{\left(-\frac{1}{2}+1\right)\times 2}\right]_0^1 = \left[\frac{(2x+3)^{1/2}}{\frac{1}{2}\times 2}\right]_0^1 = 5^{1/2} - 3^{1/2} = \sqrt{5} - \sqrt{3}$$

Short Answer Questions-I (OIQ)

[2 Mark]

Q.1. Evaluate : $\int \frac{\sin x}{\sin (x+a)} \mathrm{dx}$

Ans.

Let $x + a = t \Rightarrow dx = dt$

$\therefore \int \frac{\sin x}{\sin (x+a)} = \int \frac{\sin (t-a)}{\sin t} \mathrm{dt}$

$= \int \frac{\sin t.\ \cos a - \cos t.\ \sin a}{\sin t} \mathrm{dt}$

$= \int \cos a\mathrm{dt} - \int \cot t.\ \sin a\, \mathrm{dt} = \cos a \int \mathrm{dt} - \sin a \int \cot t\ \mathrm{dt}$

$= t.\ \cos a - \sin a.\ \log|\sin t| + C = (x + a)\ \cos a - \sin a.\ \log|\sin (x + a)| + C$

$= x \cos a + a \cos a - \sin a \log|\sin (x + a)| + C$

$= x \cos a - \sin a \log|\sin (x + a)| + C'\ [C' = C + a \cos a]$

Q.2. Evaluate : $\int_0^{\pi/2} \log (\tan x) \mathrm{dx}$

Ans.

$I = \int_0^{\pi/2} \log (\tan x) \mathrm{dx}\ \ldots (i)$

$= \int_0^{\pi/2} \log (\tan (\pi/2 - x) \mathrm{dx}$

$I = \int_0^{\pi/2} \log (\cot x) \mathrm{dx} \ldots (ii)$

Adding (i) and (ii)

$2I = \int_0^{\pi/2} \{\log (\tan x) + \log (\cot x)\} \mathrm{dx}$

$= \int_0^{\pi/2} \log (\tan x. \cot x) \mathrm{dx} = \int_0^{\pi/2} \log 1\ \ \mathrm{dx}$

$= 0. \int_0^{\pi/2} \mathrm{dx} = 0$

Q.3. Evaluate : $\int_{-1}^{1} 5x^4 \sqrt{x^5+1}\, dx$

Ans.

Let $I = \int_{-1}^{1} 5x^4 \sqrt{x^5+1}\, dx$

Let $x^5 + 1 = t \Rightarrow 5x^4\, dx = dt$

Also when, $x = -1$ $t = 0$ and when $x = 1$ $\Rightarrow t = 2$

$\therefore \quad I = \int_0^2 \sqrt{t}\, dt$

$= \int_0^2 t^{1/2} dt = \left[\frac{t^{1/2+1}}{1/2+1}\right]_0^2 = \frac{2}{3}(2^{3/2} - 0) = \frac{2}{3} 2\sqrt{2} = \frac{4\sqrt{2}}{3}$

Q.4. Let *f*(*x*) = *x* – [*x*], for every real number *x*, where [*x*] is the greatest integer less than or equal to *x*.

Evaluate: $\int_{-1}^{1} f(x)\, dx.$

Ans.

We have $f(x) = x - [x]$.

$\therefore \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} (x - [x])\, dx + \int_{0}^{1} (x - [x])\, dx$

$= \int_{-1}^{0} (x+1) dx + \int_{0}^{1} (x - 0)\, dx$

$\because$ (When $x \in (-1, 0)$; $[x] = -1$; when $x \in (0, 1)$; $[x] = -1$)

$= \left[\frac{x^2}{2} + x\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1$

Long Answer Questions-I (PYQ)

[4 Mark]

Q.1. Evaluate: $\int \frac{2x}{(x^2+1)(x^2+3)} dx$

Ans.

Let $x^2 = z \Rightarrow 2x\, dx = dz$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)} dx = \int \frac{dz}{(z+1)(z+3)}$$

Now, $\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} \quad \ldots(i)$

$$\frac{1}{(z+1)(z+3)} = \frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

$$\Rightarrow \quad 1 = A(z+3) + B(z+1) \quad \Rightarrow \quad 1 = (A+B)z + (3A+B)$$

Equating the coefficient of z and constant, we get

$$A + B = 0 \quad \ldots(ii)$$

and $\quad 3A + B = 1 \quad \ldots(iii)$

Subtracting (ii) from (iii), we get

$$2A = 1 \quad \Rightarrow \quad A = \frac{1}{2}$$

$$\therefore \quad B = -\frac{1}{2}$$

Putting the values of A and B in (i), we get

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$\therefore \int \frac{2x \ dx}{(x^2+1)(x^2+3)} = \int \frac{dz}{(z+1)(z+3)}$$

$$= \int \left(\frac{1}{2(z+1)} - \frac{1}{2(z+3)}\right) dz = \frac{1}{2}\int \frac{dz}{z+1} - \frac{1}{2}\int \frac{dz}{z+3}$$

$$= \frac{1}{2}\log|z+1| - \frac{1}{2}\log|z+3| + C = \frac{1}{2}\log|x^2+1| - \frac{1}{2}\log|x^2+3|$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

$$\begin{bmatrix}\text{Note: } \log m + \log n = \log m \, . \, n \\ \text{and } \log m - \log n = \log m/n\end{bmatrix}$$

$$= \log\sqrt{\frac{x^2+1}{x^2+3}} + C$$

Q.2. **Evaluate :** $-\int e^{2x} \sin x \, dx$

Ans.

Let $I = \int e^{2x} \sin x \ dx$

$$= -e^{2x}\cos x - \int 2e^{2x}(-\cos x)dx = -e^{2x}\cos x + 2\int e^{2x}\cos x \ dx$$

$$= -e^{2x}\cos x + 2[e^{2x}\sin x - \int 2e^{2x}\sin x \ dx]$$

$$= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x \, dx + C'$$

$$= e^{2x}(2\sin x - \cos x) - 4I + C'$$

$$\Rightarrow I = \frac{e^{2x}}{5}[2\sin x - \cos x] + C \qquad \left[\text{where } C = \frac{C'}{5}\right]$$

Q.3. **Evaluate:** $\int \sin x \sin^2 x \sin^3 x \, dx$

Ans.

Let $I = \int \sin x \sin 2x \sin 3x \ \mathrm{dx}$

$= \frac{1}{2} \int 2 \sin x . \sin 2x . \sin 3x \, \mathrm{dx} = \frac{1}{2} \int \sin x . (2 \sin 2x . \sin 3x) \, \mathrm{dx}$

$= \frac{1}{2} \int \sin x . (\cos x - \cos 5x) \, \mathrm{dx}$ $\quad [\because 2\sin A \sin B = \cos (A - B) - \cos (A + B)]$

$= \frac{1}{2\times2} \int 2 \sin x . \cos x \, \mathrm{dx} - \frac{1}{2\times2} \int 2 \sin x . \cos 5x \ \mathrm{dx}$

$= \frac{1}{4} \int \sin 2x \, \mathrm{dx} - \frac{1}{4} \int (\sin 6x - \sin 4x) \mathrm{dx}$
$\left[\because \sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}\right]$

$= -\frac{\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$

Q.4. **Evaluate:** $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x . \cos^2 x} \mathrm{dx}$

Ans.

Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x . \cos^2 x} \mathrm{dx} \quad \Rightarrow \quad I = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x . \cos^2 x} \mathrm{dx}$

$\Rightarrow \quad I = \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x . \cos^2 x + \cos^4 x)}{\sin^2 x . \cos^2 x} \mathrm{dx}$

$\Rightarrow \quad I = \int \frac{\sin^4 x - \sin^2 x . \cos^2 x + \cos^4 x}{\sin^2 x . \cos^2 x} \mathrm{dx} = \int \tan^2 x \, \mathrm{dx} - \int \mathrm{dx} + \int \cot^2 \mathrm{xdx}$

$\Rightarrow \quad I = \int (\sec^2 x - 1) \mathrm{dx} - x + \int (\mathrm{cosec}^2 x - 1) \mathrm{dx}$

$\Rightarrow I = \int \sec^2 x \, \mathrm{dx} + \int \mathrm{cosec}^2 x \, \mathrm{dx} - x - x - x + C = \tan x - \cot x - 3x + C$

Q.5. **Evaluate:** $\int (x - 3) \sqrt{x^2 + 3x - 18} \, \mathrm{dx}$

Ans.

Let $\quad I = \int (x - 3) \sqrt{x^2 + 3x - 18} \, \mathrm{dx} \quad \ldots (i)$

$x - 3 = A \frac{d}{dx} (x^2 + 3x - 18) + B \quad \Rightarrow \quad x - 3 = A(2x + 3) + B \ \ldots (ii)$

$\Rightarrow \ x - 3 = 2Ax + (3A + B)$

Equating the co-efficient, we get

$2A = 1$ and $3A + B = -3 \quad \Rightarrow \; A = \frac{1}{2}$ and $3 \times \frac{1}{2} + B = -3$

$\Rightarrow \quad A = \frac{1}{2}$ and $B = -3 - \frac{3}{2} = -\frac{9}{2}$

$\therefore \quad I \int \left(\frac{1}{2}(2x+3) - \frac{9}{2}\right) \int \sqrt{x^2 + 3x - 18}\,\mathrm{dx}$ [From (i) and (ii)]

$$I = \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\,\mathrm{dx} - \frac{9}{2}\int \sqrt{x^2+3x-18}\,\mathrm{dx}$$

$\Rightarrow \; I = \frac{1}{2}I_1 - \frac{9}{2}I_2 \; \ldots(iii)$

$$\left[\begin{array}{l}\text{where } I_1 = \int (2x+3)\sqrt{x^2+3x-18}\,\mathrm{dx} \\ \text{and } \quad I_2 = \int \sqrt{x^2+3x-18}\,\mathrm{dx}\end{array}\right]$$

Now, $I_1 = \int (2x+3)\sqrt{x^2+3x-18}\;\mathrm{dx}$

Put $x^2 + 3x - 18 = z \qquad \Rightarrow \qquad (2x+3)\,dx = dz$

$I_1 = \int \sqrt{z}\,\mathrm{dz} = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1 = \frac{2}{3}(z)^{\frac{3}{2}} + C_1 \quad \Rightarrow \quad I_1 = \frac{2}{3}(x^2+3x-18)^{\frac{3}{2}} + C_1 \ldots \; (iv)$

Again, $I_2 = \int \sqrt{x^2 + 3x - 18}\,\mathrm{dx}$

$= \int \sqrt{x^2 + 2.x.\frac{3}{2} + \left(\frac{3}{2}\right)^2 - \frac{9}{4} - 18}\mathrm{dx} = \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}$

$I_2 = \frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} - \frac{81}{4\times 2}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x-18}\right|$

$\Rightarrow I_2 = \frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} - \frac{81}{8}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x-18}\right| + C_2 \ldots \; (v)$

Putting the value of I_1 and I_2 in (iii), we get

$I = \frac{1}{3}(x^2+3x-18)^{\frac{3}{2}} - \frac{9}{4}\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} + \frac{729}{16}\log\left|\left(x+\frac{3}{2}\right) + \sqrt{x^2+3x-}\right.$

$\left[\text{where } C = \frac{C_1}{2} - \frac{9}{2}C_2\right]$

Q.6. **Evaluate:** $\int \frac{2}{(1-x)(1+x^2)}dx$

Ans.

Here $I = \int \frac{2}{(1-x)(1+x^2)}dx$

Now, $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2)+(Bx+C)(1-x)}{(1-x)(1+x^2)}$

$\Rightarrow 2 = A(1+x^2) + (Bx+C)(1-x) = A + Ax^2 + Bx - Bx^2 + C - Cx$

$\Rightarrow 2 = (A+C) + (A-B)x^2 + (B-C)x$

Equating co-efficient both sides, we get

$A + C = 2 \quad ...(i)$

$A - B = 0 \quad ...(ii)$

$B - C = 0 \quad ...(iii)$

From (ii) and (iii) $A = B = C$

Putting $C = A$ in (i), we get

$A + A = 2 \quad \Rightarrow \quad 2A = 2 \quad \Rightarrow \quad A = 1$

i.e., $A = B = C = 1$

$\therefore \quad \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$

$\therefore \quad \int \frac{2}{(1-x)(1+x^2)} = \int \frac{1}{1-x}dx + \int \frac{x+1}{1+x^2}dx = -\log|1-x| + \int \frac{x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$

$= -\log|1-x| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + C_1$

Q.7. **Find:** $\int \frac{(5x-2)}{(3x^2+2x+1)}dx$

Ans.

The given integral is in the form of $\int \frac{(\mathrm{px}+q)}{\mathrm{ax}^2+\mathrm{bx}+c} \mathrm{dx}$

Therefore, we express

$$(5x-2) = A\frac{d}{dx}(1+2x+3x^2) + B = A(2+6x) + B$$

Equating the coefficients of x and the constant term on both the sides, we get

$$6A = 5 \text{ and } 2A + B = -2 \quad \text{or} \quad A = \frac{5}{6} \text{ and } B = -\frac{11}{3}$$

$$\therefore \int \frac{(5x-2)}{(1+2x+3x^2)} \mathrm{dx} = \frac{5}{6}\int \frac{2+6x}{(1+2x+3x^2)} \mathrm{dx} - \frac{11}{3}\int \frac{\mathrm{dx}}{3x^2+2x+1}$$

$$= \frac{5}{6} I_1 - \frac{11}{3} I_2 \quad \text{(say)} \qquad \ldots(i)$$

In I_1 putting $1 + 2x + 3x^2 = t$, so that $(2 + 6x)\,dx = dt$

$$\therefore I_1 = \int \frac{\mathrm{dt}}{t} = \log|t| + C_1 = \log|3x^2 + 2x + 1| + C_1 \quad \ldots(ii)$$

and $I_2 = \int \frac{\mathrm{dx}}{3x^2+2x+1} = \int \frac{\mathrm{dx}}{3\left(x^2+\frac{2x}{3}+\frac{1}{3}\right)} = \frac{1}{3}\int \frac{\mathrm{dx}}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$

Putting , $\left(x+\frac{1}{3}\right) = t$ so that $dx = dt$, we get

$$I_2 = \frac{1}{3}\int \frac{\mathrm{dt}}{t^2+\left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{3.\frac{\sqrt{2}}{3}} \tan^{-1}\frac{t}{\frac{\sqrt{2}}{3}} + C_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\frac{3t}{\sqrt{2}} + C_2 = \frac{1}{\sqrt{2}} \tan^{-1}\frac{3\left(x+\frac{1}{3}\right)}{\sqrt{2}} + C_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\frac{(3x+1)}{\sqrt{2}} + C_2 \qquad \ldots(iii)$$

Using (ii) and (iii) in (i), we get

$$\int \frac{(5x-2)}{(1+2x+3x^2)} .\mathrm{dx} = \frac{5}{6}\log|3x^2 + 2x + 1| - \frac{11}{3\sqrt{2}} \tan^{-1}\frac{(3x+1)}{\sqrt{2}} + C$$

where, $C = C_1 + C_2$

Q.8. Evaluate: $\int \frac{3x-1}{(x+2)^2}dx$

Ans.

Let $\frac{3x-1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} \quad \Rightarrow \quad \frac{3x-1}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$

$\Rightarrow 3x - 1 = A(x+2) + B \quad \Rightarrow \quad 3x - 1 = Ax + (2A + B)$

Equating the coefficient of x and constant term both side, we get

$A = 3,\ 2A + B = -1$

$\Rightarrow 2 \times 3 + B = -1 \quad \Rightarrow \quad B = -7$

$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{x+2} - \frac{7}{(x+2)^2}$

$\Rightarrow \int \frac{3x-1}{(x+2)^2}dx = \int \frac{3}{x+2}dx - \int \frac{7}{(x+2)^2}dx$

$= 3\log\left|x+2\right| - 7\frac{(x+2)^{-1}}{-1} + C = 3\log\left|x+2\right| + \frac{7}{(x+2)} + C$

Q.9. Evaluate: $\int \frac{\sin(x-a)}{\sin(x+a)}dx$

Ans.

Let $I = \int \frac{\sin(x-a)}{\sin(x+a)}dx$

Let $x + a = t \quad \Rightarrow x = t - a \quad \Rightarrow dx = dt$

$\therefore \quad I = \int \frac{\sin(t-2a)}{\sin t}dt = \int \frac{\sin t.\cos 2a - \cos t.\sin 2a}{\sin t}dt$

$= \cos 2a \int dt - \int \sin 2a.\cot t\ dt = \cos 2a.t - \sin 2a.\log|\sin t| + C$

$= \cos 2a.(x+a) - \sin 2a.\log|\sin(x+a)| + C$

$= x\cos 2a + a\cos 2a - (\sin 2a)\log|\sin(x+a)| + C$

Q.10. Evaluate: $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$

Ans.

Let $I = \int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$

Put $e^x = t \Rightarrow e^x\, dx = dt$, we get

$$\therefore \quad I = \int \frac{dt}{\sqrt{5-4t-t^2}} = \int \frac{dt}{\sqrt{-(t^2+4t-5)}} = \int \frac{dt}{\sqrt{-(t^2+2.t.2+2^2-9)}}$$

$$= \int \frac{dt}{\sqrt{3^2-(t+2)^2}} = \sin^{-1} \frac{t+2}{3} + C = \sin^{-1} \left(\frac{e^x+2}{3}\right) + C$$

Q.11. Evaluate: $\int x \sin^{-1} x\, dx$

Ans.

Let $I = \int x \sin^{-1} x\, dx$

$$= \sin^{-1} x . \frac{x^2}{2} - \int \frac{x^2}{2\sqrt{1-x^2}} dx \qquad \text{[By using integration by part]}$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x\right] + C$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x + C$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + C$$

Q.12. Evaluate: $\int e^x \left(\frac{\sin 4x-4}{1-\cos 4x}\right) dx$

Ans.

Let $I=\int e^{x}\left(\frac{\sin 4x-4}{1-\cos 4x}\right)dx$

$=\int e^{x}\left(\frac{2\sin 2x.\cos 2x-4}{2\sin^{2}2x}\right)dx$ $\quad[\because \sin 2x=2\sin x.\cos x$ and $\cos 2x=1-\sin^{2}x]$

$=\int e^{x}(\cot 2x-2\operatorname{cosec}^{2}2x)dx$

Let $\quad f(x)=\cot 2x \therefore \quad f'(x)=-2\operatorname{cosec}^{2}2x$

$\therefore I=\int e^{x}(f(x)+f'(x))\,dx$

$\Rightarrow I=e^{x}.f(x)+C=e^{x}.\cot 2x+C$ $\quad[\because \int e^{x}(f(x)+f'(x))dx=e^{x}f(x)+C]$

Q.13. **Evaluate:** $\int\frac{x+2}{\sqrt{x^{2}+5x+6}}dx$

Ans.

Let $I=\int\frac{x+2}{\sqrt{x^{2}+5x+6}}dx$

Now, we can express as

$x+2=A\frac{d}{dx}(x^{2}+5x+6)+B$

$\Rightarrow x+2=A(2x+5)+B \quad\Rightarrow\quad x+2=2Ax+(5A+B)$

Equating coefficients both sides, we get

$2A=1,\ 5A+B=2 \quad\Rightarrow\quad A=\frac{1}{2},B=2-\frac{5}{2}=-\frac{1}{2}$

$\therefore\ x+2=\frac{1}{2}(2x+5)-\frac{1}{2}$

Hence, $\quad I=\int\frac{\frac{1}{2}(2x+5)-\frac{1}{2}}{\sqrt{x^{2}+5x+6}}dx=\frac{1}{2}\int\frac{2x+5}{\sqrt{x^{2}+5x+6}}dx-\frac{1}{2}\int\frac{dx}{\sqrt{x^{2}+5x+6}}$

$I = \frac{1}{2} . I_1 - \frac{1}{2} I_2 \quad \dots (i)$

where, $I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}} dx , I_2 = \int \frac{dx}{\sqrt{x^2+5x+6}}$

Now, $I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}} dx ,$

Let $x^2 + 5x + 6 = z \quad \Rightarrow \quad (2x + 5) dx = dz$

$\therefore \; I_1 = \int \frac{dz}{\sqrt{z}} = \int z^{\frac{-1}{2}} dz = \frac{z^{-\frac{1}{2}+1}}{\frac{-1}{2}+1} + C_1 = 2\sqrt{z} + C_1 = 2\sqrt{x^2 + 5x + 6} + C_1$

Again $I_2 = \int \frac{dx}{\sqrt{x^2+5x+6}} = \int \frac{dx}{\sqrt{x^2+2\times x \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \frac{25}{4} + 6}}$

$= \int \frac{dx}{\left(x+\frac{5}{2}\right)^2 - \frac{1}{4}} = \int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$

$= \log \left| \left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6} \right| + C_2$

Putting the value of I_1 and I_2 in (i), we get

$I = \frac{1}{2} \left\{ 2\sqrt{x^2 + 5x + 6} + C_1 \right\} - \frac{1}{2} \left\{ \log \left| \left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6} \right| + C_2 \right\}$

$\Rightarrow \; I = \sqrt{x^2 + 5x + 6} - \frac{1}{2} \log \left| \left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6} \right| + \frac{1}{2} C_1 - \frac{1}{2} C_2$

$= \sqrt{x^2 + 5x + 6} - \frac{1}{2} \log \left| \left(x + \frac{5}{2}\right) + \sqrt{x^2 + 5x + 6} \right| + C$

[where $C = \frac{1}{2} C_1 - \frac{1}{2} C_2$]

Q.14. **Evaluate:** $\int \frac{(x^2 - 3x)}{(x-1)(x-2)} dx$

Ans.

Let $I=\int \frac{(x^2-3x)}{(x-1)(x-2)}dx=\int \frac{(x^2-3x)}{x^2-3x+2}dx$

$=\int \frac{x^2-3x+2-2}{x^2-3x+2}dx=\int dx-\int \frac{2\,dx}{x^2-3x+2}$

$=x-2\int \frac{dx}{x^2-2.x.\frac{3}{2}+\frac{9}{4}-\frac{9}{4}+2}=x-2\int \frac{dx}{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$

$=x-2\log\left|\frac{x-\frac{3}{2}-\frac{1}{2}}{x-\frac{3}{2}+\frac{1}{2}}\right|+C$

$=x-2\log\left|\frac{x-2}{x-1}\right|+C \quad \left[\because \int \frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$

Q.15. **Evaluate:** $\int x^2\tan^{-1}x\,dx$

Ans.

$\int x^2\tan^{-1}x\,dx=\tan^{-1}x.\frac{x^3}{3}-\int \frac{1}{1+x^2}.\frac{x^3}{3}dx$ [By using integration by parts]

$=\frac{x^3\tan^{-1}x}{3}-\frac{1}{3}\int\left(x-\frac{x}{x^2+1}\right)dx$

$$\left[\begin{array}{l} 1+x^2 \overset{x}{\overline{\big)x^3}} \\ \qquad \frac{-x^3\pm x}{-x} \end{array}\right]$$

$=\frac{x^3\tan^{-1}x}{3}-\frac{1}{3}\left[\int x dx-\int\frac{x}{x^2+1}dx\right]$

$=\frac{x^3\tan^{-1}x}{3}-\frac{1}{3}\frac{x^2}{2}+\frac{1}{3}\int\frac{dz}{2z}$

$$\left[\begin{array}{l} Let\ \ x^2+1=z \\ \Rightarrow\ \ 2xdx=dz \\ \Rightarrow\ \ \ xdx=\frac{dz}{2} \end{array}\right]$$

$=\frac{x^3\tan^{-1}x}{3}-\frac{x^2}{6}+\frac{1}{6}\log|z|+C$

$=\frac{x^3\tan^{-1}x}{3}-\frac{x^2}{6}+\frac{1}{6}\log\left|x^2+1\right|+C$

Q.16. Evaluate: $\int \frac{x^2+1}{(x-1)^2\,(x+3)}dx$

Ans.

Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$

$\Rightarrow \quad \frac{x^2+1}{(x-1)^2(x+3)} = \frac{A(x-1)\,(x+3)+B(x+3)+C(x-1)^2}{(x-1)^2(x+3)}$

$\Rightarrow \; x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2 \qquad \ldots(i)$

Putting $x = 1$ in (i), we get

$\Rightarrow \; 2 = 4B \quad \Rightarrow \quad B = \frac{1}{2}$

Putting $x = -3$ in (i), we get

$\Rightarrow 10 = 16C \quad \Rightarrow \qquad C = \frac{10}{16} = \frac{5}{8}$

Putting $x = 0,\, B = \frac{1}{2}, C = \frac{5}{8}$ in (i), we get

$1 = A(-1)\,.\,(3) + \frac{1}{2} \times 3 + \frac{5}{8}(-1)^2 \qquad \Rightarrow \qquad 1 = -3A + \frac{3}{2} + \frac{5}{8}$

$\Rightarrow \; 3A = \frac{12+5}{8} - 1 = \frac{17}{8} - 1 = \frac{9}{8} \qquad \Rightarrow \qquad A = \frac{3}{8}$

$\therefore \; \frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8(x-1)} + \frac{1}{2(x-1)^2} + \frac{5}{8(x+3)}$

$\therefore \; \int \frac{x^2+1}{(x-1)^2(x+3)}dx = \int \left(\frac{3}{8(x-1)} + \frac{1}{2(x-1)^2} + \frac{5}{8(x+3)} \right) dx$

$= \frac{3}{8} \int \frac{dx}{x-1} + \frac{1}{2} \int (x-1)^{-2}dx + \frac{5}{8} \int \frac{dx}{x+3}$

$= \frac{3}{8}\log\left|x-1\right| - \frac{1}{2(x-1)} + \frac{5}{8}\log\left|x+3\right| + C$

Q.17. Find : $\int \frac{dx}{\sin x+\sin 2x}$

Ans.

Here, $I = \int \frac{1}{\sin x+\sin 2x} dx$

$\Rightarrow \quad I = \int \frac{1}{\sin x+2\sin x\cos x} dx \qquad \Rightarrow I = \int \frac{1}{\sin x(1+2\cos x)} dx$

$\Rightarrow \quad I = \int \frac{\sin x}{\sin^2 x(1+2\cos x)} dx \qquad \Rightarrow I = \int \frac{\sin x}{(1-\cos^2 x)(1+2\cos x)} dx$

Let $\cos x = z \quad \Rightarrow \; -\sin x\, dx = dz$

$\Rightarrow \; I = \int \frac{-dz}{(1-z^2)(1+2z)} \qquad \Rightarrow \qquad I = -\int \frac{dz}{(1+z)(1-z)(1+2z)}$

Here, integrand is proper rational function. Therefore, by the form of partial function, we can write

$\frac{1}{(1+z)(1-z)(1+2z)} = \frac{A}{1+z} + \frac{B}{1-z} + \frac{C}{1+2z} \qquad ...(i)$

$\Rightarrow \quad \frac{1}{(1+z)(1-z)(1+2z)} = \frac{A(1-z)(1+2z)+B(1+z)(1+2z)+C(1+z)(1-z)}{(1+z)(1-z)(1+2z)}$

$\Rightarrow \; 1 = A(1-z)(1+2z)+B(1+z)(1+z)(1+2z) + C(1+z)(1-z) \qquad ....(ii)$

Putting the value of $z = -1$ in (ii), we get

$\Rightarrow 1 = -2A + 0 + 0 \qquad \Rightarrow \qquad A = -1/2$

Again, putting the value of $z = 1$ in (ii), we get

$\Rightarrow \; 1 = 0 + B.2.(1+2) + 0 \Rightarrow 1 = 6B \Rightarrow B = \frac{1}{2}$

Similarly, putting the value of $z = -\frac{1}{2}$ in (ii), we get

$\Rightarrow \; 1 = 0 + 0 + C\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \qquad \Rightarrow \qquad 1 = \frac{3}{4}C \qquad \Rightarrow \qquad C = \frac{4}{3}$

Putting the value of A, B, C in (i), we get

$$\frac{1}{(1+z)(1-z)(1-2z)}=\frac{1}{2(1+z)}+\frac{1}{6(1-z)}+\frac{4}{3(1+2z)}$$

$$\therefore\ I=-\int\left[-\frac{1}{2(1+z)}+\frac{1}{6(1-z)}+\frac{4}{3(1+2z)}\right]\mathrm{dz}$$

$$\therefore\ I=\int\left[\frac{1}{2(1+z)}-\frac{1}{6(1-z)}-\frac{4}{3(1+2z)}\right]\mathrm{dz}$$

$$\Rightarrow\ I=\frac{1}{2}\log|1+z|+\frac{1}{6}\log|1-z|-\frac{4}{3\times 2}\log|1+2z|+C$$

Putting the value of z, we get

$$\Rightarrow\ I=\frac{1}{2}\log|1+\cos x|+\frac{1}{6}\log|1-\cos x|-\frac{2}{3}\log|1+2\cos x|+C$$

Q.18. Integrate the following w.r.t. *x*.

$$\frac{x^2-3x+1}{\sqrt{1-x^2}}$$

Ans.

$$\text{Let } I=\int\frac{x^2-3x+1}{\sqrt{1-x^2}}\mathrm{dx}=\int\frac{x^2-1+2-3x}{\sqrt{1-x^2}}\mathrm{dx}$$

$$=\int\frac{-(1-x^2)}{\sqrt{1-x^2}}\mathrm{dx}+2\int\frac{\mathrm{dx}}{\sqrt{1-x^2}}-3\int\frac{\mathrm{xdx}}{\sqrt{1-x^2}}$$

$$=-\int\sqrt{1-x^2}\,\mathrm{dx}+2\int\frac{\mathrm{dx}}{\sqrt{1-x^2}}-3\int\frac{\mathrm{xdx}}{\sqrt{1-x^2}}$$

$$=-\frac{1}{2}x\sqrt{1-x^2}-\frac{1}{2}\sin^{-1}x+2\sin^{-1}x+3\sqrt{1-x^2}+C$$

$$=\frac{3}{2}\sin^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+3\sqrt{1-x^2}+C$$

Q.19. Evaluate: $\int\frac{x^2+1}{(x^2+4)(x^2+25)}\mathrm{dx}$

Ans.

Let $I=\int \frac{x^2+1}{(x^2+4)(x^2+25)}\mathrm{dx}$

Put $x^2 = y \quad \Rightarrow \quad \frac{x^2+1}{(x^2+4)(x^2+25)}=\frac{y+1}{(y+4)(y+25)}$

Now, $\frac{y+1}{(y+4)(y+25)}=\frac{A}{y+4}+\frac{B}{y+25}$

$\Rightarrow \quad \frac{y+1}{(y+4)(y+5)}=\frac{A(y+25)+B(y+4)}{(y+4)(y+5)}$

$\Rightarrow y+1=(A+B)y+(25A+4B)$

Equating coefficients, we get

$A+B=1$ and $25A+4B=1$

$\Rightarrow A=\frac{-1}{7}, B=\frac{8}{7}$

$\therefore \frac{x^2+1}{(x^2+4)(x^2+25)}=\frac{-1}{7(x^2+4)}+\frac{8}{7(x^2+25)}$

$\therefore I=\int\left[-\frac{1}{7(x^2+4)}+\frac{8}{7(x^2+25)}\right]\mathrm{dx}=-\frac{1}{7}\int\frac{\mathrm{dx}}{x^2+2^2}+\frac{8}{7}\int\frac{\mathrm{dx}}{x^2+5^2}$

$=-\frac{1}{7}\times\frac{1}{2}\tan^{-1}\frac{x}{2}+\frac{8}{7}\times\frac{1}{5}\tan^{-1}\frac{x}{5}+C$

$=-\frac{1}{14}\tan^{-1}\frac{x}{2}+\frac{8}{35}\tan^{-1}\frac{x}{5}+C$

Q.20. **Evaluate:** $\int\frac{\mathrm{dx}}{x(x^5+3)}$

Ans.

Let $I=\int \frac{dx}{x(x^5+3)}=\int \frac{x^4\,dx}{x^5(x^5+3)}=\frac{1}{5}\int \frac{5x^4\,dx}{x^5(x^5+3)}$

Put $x^5=z \quad \Rightarrow \quad 5x^4\,dx=dz$

$\therefore\ I=\frac{1}{5}\int \frac{dz}{z(z+3)}=\frac{1}{5\times 3}\int \frac{z+3-z}{z(z+3)}dz$

$=\frac{1}{15}\int \frac{z+3}{z(z+3)}dz-\frac{1}{15}\int \frac{z}{z(z+3)}dz$

$=\frac{1}{15}\int .\frac{dz}{z}-\frac{1}{15}\int \frac{dz}{z+3}=\frac{1}{15}\{\log z-\log|z+3|\}+C$

$=\frac{1}{15}\log\left|\frac{z}{z+3}\right|+C=\frac{1}{15}\log\left|\frac{x^5}{x^5+3}\right|+C$

Q.21. **Evaluate:** $\int \frac{2x^2+3}{x^2+5x+6}dx$

Ans.

Let $I=\int \frac{2x^2+3}{x^2+5x+6}dx$

$=\int\left(2-\frac{10x+9}{x^2+5x+6}\right)dx=2\int dx-\int \frac{10x+9}{x^2+5x+6}dx$

$$[x^2+5x+6\overline{\big)2x^2+3}^{\;2}]$$

$$\frac{-2x^2\pm 10x\pm 12}{-10x-9}$$

$=2x-\int \frac{10x+9}{x^2+3x+2x+6}dx$

$=2x-\int \frac{10x+9}{x(x+3)+2(x+3)}dx=2x-\int \frac{10x+9}{(x+3)(x+2)}dx$

$= 2x - \int \left(\frac{-11}{x+2} + \frac{21}{x+3} \right) dx$

$$\left[\begin{array}{l} \frac{10x+9}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \\ \Rightarrow \quad 10x + 9 = A(x+3) + B(x+2) \\ \text{Putting } x = -3, \text{ we get } B = 21 \\ \text{Putting } x = -2, \text{ we get } A = -11 \end{array} \right]$$

$= 2x + 11 \int \frac{dx}{x+2} - 21 \int \frac{dx}{x+3}$

$= 2x + 11 \log |x+2| - 21 \log |x+3| + C$

Q.22. Evaluate : $\int (3x - 2)\sqrt{x^2 + x + 1}\, dx$

Ans.

Let $I = \int (3x - 2) \sqrt{x^2 + x + 1}\ dx$

Let $3x - 2 = A \frac{d}{dx} (x^2 + x + 1) + B$

$\Rightarrow 3x - 2 = A (2x + 1) + B$

$\Rightarrow 3x - 2 = 2Ax + (A + B)$

Equating we get

$2A = 3$ and $A + B = -2$

$A = \frac{3}{2}$ and $B = -2 - \frac{3}{2} = -\frac{7}{2}$

Now, $I = \int \left\{ \frac{3}{2} (2x + 1) - \frac{7}{2} \right\} \sqrt{x^2 + x + 1}\ dx$

$= \frac{3}{2} \int (2x + 1) \sqrt{x^2 + x + 1}\ dx - \frac{7}{2} \int \sqrt{x^2 + x + 1}\ dx$

$\Rightarrow I = \frac{3}{2} I_1 - \frac{7}{2} I_2 \quad \ldots (i)$

Where, $I_1 = \int (2x+1)\sqrt{x^2+x+1}\ \mathrm{dx}$ and $I_2 = \sqrt{x^2+x+1}\ \mathrm{dx}$

Now, $I_1 = \int (2x+1)\sqrt{x^2+x+1}\ \mathrm{dx}$

Let $x^2 + x + 1 = z \quad \Rightarrow (2x+1)\ dx = dz$

$\Rightarrow I_1 = \int \sqrt{z}\ \mathrm{dz}$

$= \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1 = \frac{2}{3} z^{3/2} + C_1$

$I_1 = \frac{2}{3}(x^2+x+1)^{3/2} + C_1 \quad \ldots (ii)$

Again $I_2 = \int \sqrt{x^2+x+1}\ \mathrm{dx}$

$= \int \sqrt{x^2 + 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{4} + 1}\,\mathrm{dx}$

$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{dx}$

$I_2 = \int \sqrt{x^2+x+1}\ dx$

$I_2 = \frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 .\ \log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + C_2 \quad \ldots (iii)$

Putting value of I_1 and I_2 from (ii), (iii) in (i), we get

$I = (x^2+x+1)^{3/2} - \frac{7}{4}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} - \frac{21}{16}\log\ \left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| +$

[where $C = \frac{3}{2}C_1 - \frac{7}{2}C_2$]

Q.23. **Evaluate :** $\int \frac{x^2}{(x^2+4)(x^2+9)}\mathrm{dx}$

Ans.

Let $I=\int \frac{x^2}{(x^2+4)(x^2+9)}\mathrm{dx}$

Put $x^2 = t$, we get

$\therefore \quad \frac{x^2}{(x^2+4)(x^2+9)} = \frac{t}{(t+4)(t+9)}$

Now, $\frac{t}{(t+4)(t+9)} = \frac{A}{t+4} + \frac{B}{t+9} = \frac{A(t+9)+B(t+4)}{(t+4)(t+9)}$

$\Rightarrow t = (A+B)t + (9A+4B)$

Equating the coefficients, we get

$A + B = 1,\ 9A + 4B = 0$

Solving above two equations, we get

$A = -\frac{4}{5},\ B = \frac{9}{5}$

$\therefore \frac{x^2}{(x^2+4)(x^2+9)} = -\frac{4}{5(x^2+4)} + \frac{9}{5(x^2+9)}$

$=-\frac{4}{5} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{9}{5} \times \frac{1}{3}\tan^{-1}\frac{x}{3} + C$

$I = -\frac{4}{5}\int \frac{\mathrm{dx}}{x^2+2^2} + \frac{9}{5}\int \frac{\mathrm{dx}}{x^2+3^2} = \frac{4}{5} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{9}{5} \times \frac{1}{3}\tan^{-1}\frac{x}{3} + C$

$= -\frac{2}{5}\tan^{-1}\frac{x}{2} + \frac{3}{5}\tan^{-1}\frac{x}{3} + C$

Q.24. **Find:** $\int \frac{(3\sin\theta - 2)\cos\theta}{5-\cos^2\theta - 4\sin\theta}d\theta$

Ans.

We have

$$I = \int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$$

Let $\sin\theta = z$

$\Rightarrow \cos\theta d\theta = dz$

$$\therefore\ I = \int \frac{(3z-2)\,dz}{5-(1-z^2)-4z}$$

$$= \int \frac{(3z-2)\,dz}{5-1+z^2-4z} = \int \frac{(3z-2)}{4-4z+z^2} dz$$

$$= \int \frac{3z-2}{(z-2)^2} dz = \int \frac{3z}{(z-2)^2} dz - 2\int \frac{dz}{(z-2)^2}$$

Let $z - 2 = t \quad \Rightarrow \quad dz = dt$

$$= \int \frac{3(t+2)\,dt}{t^2} - 2\int \frac{dt}{t^2} = 3\int \frac{t\,.\,dt}{t^2} + 6\int \frac{dt}{t^2} - 2\int \frac{dt}{t^2}$$

$$= 3\int \frac{dt}{t} + 4\int \frac{dt}{t^2} = 3\log|t| + 4\frac{t^{-2+1}}{-2+1} + C$$

$$= 3\log|t| - 4.\frac{1}{t} + C$$

Putting value of t in terms of z then z in terms of q, we get

$$= 3\log\left|\sin\theta - 2\right| - \frac{4}{\sin\theta - 2} + C$$

Q.25. **Find:** $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$

Ans.

We have

$$I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$$

$$= \int \frac{x^{1/2}\,dx}{\sqrt{a^3 - x^3}}$$

Let $x^{3/2}=t \Rightarrow \frac{3}{2}x^{1/2}dx=dt \Rightarrow x^{1/2}dx=\frac{2}{3}dt$

$I=\frac{2}{3}\int\frac{dt}{\sqrt{(a^{3/2})^2-t^2}}$ $\left[\because x^{3/2}=t \Rightarrow x^3=t^2\right]$

$=\frac{2}{3}\sin^{-1}\left(\frac{t}{a^{3/2}}\right)+C$

$\Rightarrow \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right)+C$

$=\frac{2}{3}\sin^{-1}\sqrt{\frac{x^3}{a^3}}+C$

Q.26. **Find:** $\int\frac{(2x-5)e^{2x}}{(2x-3)^3}dx$

Ans.

We have,

$I=\int\frac{(2x-5)e^{2x}}{(2x-3)^3}dx$

$=\int e^{2x}\left[\frac{(2x-3)-2}{(2x-3)^3}\right]dx$

$$\Rightarrow\ e^3\int e^{2x-3}\left[\frac{1}{(2x-3)^2}-\frac{2}{(2x-3)^3}\right]\mathrm{dx}$$

Let $2x - 3 = t$

$$\Rightarrow\ 2dx = dt\ \Rightarrow\ dx = \frac{dt}{2}$$

$$\Rightarrow\ I = \frac{e^3}{2}\int e^t\left[\frac{1}{t^2}-\frac{2}{t^3}\right]\mathrm{dt}$$

$$\Rightarrow\ I = \frac{e^3}{2}e^t.\ \frac{1}{t^2}+C$$

Putting $t = 2x - 3$

$$I = \frac{e^3}{2}e^{2x-3}\frac{1}{(2x-3)^2}+C$$

$$\Rightarrow\ I = \frac{e^{2x}}{2(2x-3)^2}+C$$

Q.27. Find : $\int(2x+5)\sqrt{10-4x-3x^2}\,\mathrm{dx}$

Ans.

Let, $I = \int(2x+5)\sqrt{10-4x-3x^2}\,\mathrm{dx}$

Let $(2x+5) = A\frac{d}{\mathrm{dx}}(10-4x-3x^2)+B$

$\Rightarrow 2x + 5 = A(-4 - 6\boldsymbol{x}) + B$

$\Rightarrow 2x + 5 = -4A - 6Ax + B$

Equating, we get

$-4A + B = 5$ ...(1) and $-6A = 2$ ...(2)

$(2)\quad \Rightarrow A = -\frac{1}{3}$

Now, from (1) $\frac{4}{3} + B = 5 \ \Rightarrow \ B = 5 - \frac{4}{3} = \frac{11}{3}$

$\therefore 2x + 5 = -\frac{1}{3}(-4 - 6x) + \frac{11}{3}$

Now, $I = \int\left\{-\frac{1}{3}(-4 - 6x) + \frac{11}{3}\right\}\sqrt{10 - 4x - 3x^2}\,\mathrm{dx}$

$= -\frac{1}{3}\int(-4 - 6x)\sqrt{10 - 4x - 3x^2}\,\mathrm{dx} + \frac{11}{3}\int\sqrt{10 - 4x - 3x^2}\,\mathrm{dx}$

$I = -\frac{1}{3}I_1 + \frac{11}{3}I_2 \qquad ...(i)$

where $I_1 = \int(-4 - 6x)\sqrt{10 - 4x - 3x^2}\,\mathrm{dx}$ and $I_2 = \int(10 - 4x - 3x^2)\,\mathrm{dx}$

Now, $I_1 = \int(-4 - 6x)\sqrt{10 - 4x - 3x^2}\,\mathrm{dx}$

Let $\quad 10 - 4x - 3x^2 = z$

$\Rightarrow (-4 - 6x)\ dx = dz$

$\therefore I_1 = \int\sqrt{z}\,\mathrm{dz} = \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1$

$= \frac{2}{3}(10 - 4x - 3x^2)^{3/2} + C_1 \qquad ...(ii)$

Again $I_2 = \int\sqrt{10 - 4x - 3x^2}\,\mathrm{dx} = \int\sqrt{-3\left(x^2 + \frac{4}{3}x - \frac{10}{3}\right)}\,\mathrm{dx}$

$= \sqrt{3}\int\sqrt{-\left\{x^2 + 2x.\frac{2}{3} + \frac{4}{9} - \frac{4}{9} - \frac{10}{3}\right\}}\mathrm{dx} = \sqrt{3}\int\sqrt{-\left\{\left(x + \frac{2}{3}\right)^2 - \frac{34}{9}\right\}}\,\mathrm{dx}$

$$= \sqrt{3} \int \sqrt{\frac{34}{9} - \left(x + \frac{2}{3}\right)^2}\, dx = \sqrt{3} \int \sqrt{\left(\frac{\sqrt{34}}{3}\right)^2 - \left(x + \frac{2}{3}\right)^2}\; dx$$

$$= \frac{\sqrt{3}}{2}\left(x + \frac{2}{3}\right)\sqrt{10 - 4x - 3x^2} + \frac{\sqrt{3}}{2} \times \frac{34}{9} \sin^{-1}\left(\frac{x + \frac{2}{3}}{\sqrt{\frac{34}{3}}}\right) + C_2 \quad ...(iii)$$

Putting the value of I_1 and I_2 in (i), we get

$$I = -\frac{1}{3} \times \frac{2}{3}(10 - 4x - 3x^2)^{3/2} + \frac{11}{2\sqrt{3}}\left(x + \frac{2}{3}\right)\sqrt{10 - 4x - 3x^2} + \frac{17}{9}\sin^{-1}\left(\frac{3}{\sqrt{34}}(x + \frac{2}{3})\right) + C$$

Q.28. **Evaluate:** $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

Ans.

Let $I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad ...(i)$

$$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx \qquad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x)\, dx\right]$$

$$I = \int_0^1 \log\left(\frac{x}{1-x}\right) dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^1 \log\left(\frac{1-x}{x}\right) dx + \int_0^1 \log\left(\frac{x}{1-x}\right) dx$$

$$= \int_0^1 \log\left(\frac{1-x}{x} \cdot \frac{x}{1-x}\right) dx \qquad [\because \log A + \log B = \log(A \times B)]$$

$$= \int_0^1 \log 1\, dx$$

$$2I = 0$$

$$\therefore\; I = 0$$

Q.29. **Evaluate:** $\int_0^\pi \frac{4x \sin x}{1 + \cos^2 x} dx$

Ans.

Let $\quad I = \int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x} dx \ldots (i)$

$$= \int_0^{\pi} \frac{4(\pi - x). \sin (\pi - x)}{1+\cos^2 (\pi - x)} dx$$

$$I = \int_0^{\pi} \frac{4(\pi - x). \sin x}{1+\cos^2 x} dx \ldots (ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{4(x+\pi-x) \sin x}{1+\cos^2 x} dx \quad \Rightarrow \quad 2I = 4\int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$$

$$I = 2\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Let $\cos x = z \quad \Rightarrow \quad -\sin x\, dx = dz \quad \Rightarrow \quad \sin x\, dx = -\, dz$

The limits are, $x = 0 \quad \Rightarrow \quad z = 1$

$x = \pi \quad \Rightarrow \quad z = -1$

$$\therefore\ I = 2\pi \int_1^{-1} \frac{-dz}{1+z^2} = 2\pi [\tan^{-1} z]_{-1}^{1}$$

$$= 2\pi [\tan^{-1} 1 - \tan^{-1}(-1)] = 2\pi \left[\frac{\pi}{4} + \frac{\pi}{4}\right] = 2\pi \times \frac{\pi}{2}$$

$\Rightarrow I = \pi^2.$

Q.30. **Evaluate:** $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$

Ans.

Here, $I = \int_{-\pi}^{\pi} (\cos\ ax - \sin\ bx)^2\, dx$

$\Rightarrow\ I = \int_{-\pi}^{\pi} (\cos^2\ ax + \sin^2\ bx - 2\ \cos\ ax\ \sin\ bx)\, dx$

$\Rightarrow\ I = \int_{-\pi}^{\pi} \cos^2\ ax\ dx + \int_{-\pi}^{\pi} \sin^2 bx\ dx - \int_{-\pi}^{\pi} 2\cos\ ax\ \sin bx\ dx$

$\Rightarrow\ I = 2\int_{0}^{\pi} \cos^2\ ax\ dx + 2\int_{0}^{\pi} \sin^2 bx\ dx - 0$ [first two integrands are even function while third is odd function.]

$\Rightarrow I = \int_{0}^{\pi} 2\cos^2\ ax\ dx + \int_{0}^{\pi} 2\ \sin^2\ bx\ dx$

$\Rightarrow\ I = \int_{0}^{\pi} (1 + \cos\ 2ax)\ dx + \int_{0}^{\pi} (1 - \cos 2bx)\, dx$

$\Rightarrow I = \int_{0}^{\pi} dx + \int_{0}^{\pi} \cos 2ax\ dx + \int_{0}^{\pi} dx - \int_{0}^{\pi} \cos 2bx\ dx$

$\Rightarrow\ I = 2\int_{0}^{\pi} dx + \int_{0}^{\pi} \cos 2\ ax\ dx - \int_{0}^{\pi} \cos\ 2\ bx\ dx$

$\Rightarrow I = 2[x]_0^{\pi} + \left[\frac{\sin 2\ ax}{2a}\right]_0^{\pi} - \left[\frac{\sin 2\ bx}{2b}\right]_0^{\pi}$

$\Rightarrow I = 2\pi + \frac{\sin 2\ a\pi}{2a} - \frac{\sin 2b\pi}{2b}$

Q.31. **Evaluate:** $\int\limits_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

Ans.

Let $I = \int\limits_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$= \int\limits_0^{\pi/2} \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right) dx = \int\limits_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\, dx$

$= \sqrt{2} \int\limits_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}}\, dx$ [Multiplying and dividing by $\sqrt{2}$]

$$\Rightarrow\ I=\sqrt{2}\int_0^{\pi/2}\frac{(\sin x+\cos x)}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Let $\sin x-\cos x=t\ \Rightarrow\ (\cos x+\sin x)\ dx=dt$

Now, when $x=0\ \Rightarrow\ t=-1$ and $x=\frac{\pi}{2}\ \Rightarrow\ t=1$

$$\therefore I=\sqrt{2}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}[\sin^{-1}t]_{-1}^{1}=\sqrt{2}[\sin^{-1}(1)-\sin^{-1}(-1)]$$

$$=\sqrt{2}[sin^{-1}(1)+\sin^{-1}(1)]=\sqrt{2}.2\sin^{-1}(1)=\sqrt{2}.2.\frac{\pi}{2}=\sqrt{2}\pi.$$

Q.32. **Evaluate:** $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$

Ans.

Let $I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx\quad\Rightarrow\quad I=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)\,dx}{1+\cos^2(\pi-x)}$

$=\int_0^{\pi}\frac{(\pi-x)\sin x\ dx}{1+\cos^2 x}=\pi\int_0^{\pi}\frac{\sin x\ dx}{1+\cos^2 x}-I$

or $2I=\pi\int_0^{\pi}\frac{\sin x\ dx}{1+\cos^2 x}$ or $I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x\ dx}{1+\cos^2 x}$

Put $\cos x=t$ so that $-\sin x\ dx=dt$.

The limits are, when $x=0$, $t=1$ and $x=$ p, $t=-1$, we get

$I=\frac{-\pi}{2}\int_1^{-1}\frac{dt}{1+t^2}=\pi\int_0^1\frac{dt}{1+t^2}\quad\left[\because\int_a^{-a}f(x)dx=-\int_{-a}^{a}f(x)\,dx\text{ and }\int_0^{2a}f(x)dx=2\int_0^{\cdot}\right.$

$=\pi[\tan^{-1}t]_0^1=\pi[\tan^{-1}1-\tan^{-1}0]=\pi\left[\frac{\pi}{4}-0\right]=\frac{\pi^2}{4}$

Q.33. **Find:** $\int_0^{\pi/4}\frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

Ans.

Let $I = \int\limits_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

$= \int\limits_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{2.2\sin x.\cos x}}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^3 x\sqrt{\frac{\sin x}{\cos x}.\cos^2 x}} = \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^4 x\sqrt{\tan x}}$

$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x \ dx}{\sqrt{\tan x}} = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2 x.\ \sec^2 x \ dx}{\sqrt{\tan x}}$

Let $\tan x = t$, $x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{4} \Rightarrow t = 1$

$\sec^2 x\, dx = dt$

$\therefore I = \frac{1}{2}\int_0^1 \frac{(1+t^2)\,dt}{\sqrt{t}}$

$= = \frac{1}{2}\int_0^1 (t^{-1/2} + t^{3/2})dt = \frac{1}{2}\left[\frac{t^{-1/2+1}}{-1/2+1}\right]_0^1 + \frac{1}{2}\left[\frac{t^{3/2+1}}{3/2+1}\right]_0^1$

$\frac{1}{2} \times \frac{2}{1}[\sqrt{t}]_0^1 + \frac{1}{2} \times \frac{2}{5}\left[t^{5/2}\right]_0^1 = 1 + \frac{1}{5} = \frac{6}{5}$

Q.34. **Evaluate:** $\int\limits_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}dx$

Ans.

Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$

$$= \int_{-\pi/2}^{0} \frac{\cos x}{1+e^x} dx + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$$

$$= \int_{\pi/2}^{0} \frac{\cos t}{1+e^{-t}} (-dt) + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$$

In 1st Integrand
Let $x = -t$
$dx = -dt$
$x = -\pi/2 \Rightarrow t = \pi/2$
$x = 0 \Rightarrow t = 0$

$$= \int_{0}^{\pi/2} \frac{\cos t}{1+\frac{1}{e^t}} dt + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx = \int_{0}^{\pi/2} \frac{e^t . \cos t}{1+e^t} dt + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$$

$$= \int_{0}^{\pi/2} \frac{e^x . \cos x}{1+e^x} dx + \int_{0}^{\pi/2} \frac{\cos x}{1+e^x} dx$$

[By property $\int_a^b f(x)dx = \int_a^b f(t)dt$]

$$= \int_{0}^{\pi/2} \frac{(e^x+1) . \cos x}{1+e^x} dx = \int_{0}^{\pi/2} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin \pi/2 - \sin 0$$

$$= 1.$$

Q.35. **Evaluate:** $\int_0^{\pi} \frac{x}{1+\sin x} dx$

Ans.

Let $I = \int_0^{\pi} \frac{x \, dx}{1+\sin x}$ $\quad [\because \int_0^a f(x) dx = \int_0^a f(a-x)\, dx]$ $\quad ...(i)$

$\therefore I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx = \pi \int_0^{\pi} \frac{dx}{1+\sin x} - \int_0^{\pi} \frac{x dx}{1+\sin x} = \pi \int_0^{\pi} \frac{dx}{1+\sin x} - I$ $\quad ...(ii)$

$\Rightarrow$

$2I = \pi \int_0^{\pi} \frac{dx}{1+\sin x} = 2\pi \int_0^{\pi/2} \frac{dx}{1+\sin x}$ $\quad [\because \int_0^{2a} f(x) dx = 2\int_0^a f(x) dx$ as $\sin(\pi - x) = \sin x$

$\Rightarrow \ I = \pi \int_0^{\pi/2} \frac{dx}{1+\sin x} = \pi \int_0^{\pi/2} \frac{dx}{1+\sin\left(\frac{\pi}{2} - x\right)}$ $\quad [\because \int_0^a f(x) dx = \int_0^a f(a-x) dx]$

$$= \pi \int_0^{\pi/2} \frac{dx}{1+\cos x} = \pi \int_0^{\pi/2} \frac{dx}{2\cos^2 x/2}$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sec^2 \frac{x}{2} dx = \frac{\pi}{2} \left[\frac{\tan x/2}{1/2}\right]_0^{\pi/2}$$

$$= \pi \left[\tan \frac{\pi}{4} - \tan 0\right] = \pi [1 - 0] = \pi$$

Q.36. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

Ans.

Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}$

[By $u \sin g$ property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$] ...(i)

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{1}{\sqrt{\tan x}}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}dx$ ...(ii)

Adding (i) and (ii), we get

$2I = \int_{\pi/6}^{\pi/3} \frac{(1+\sqrt{\tan x})}{(1+\sqrt{\tan x})}dx$

$= \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$2I = \frac{\pi}{6}$ or $I = \frac{\pi}{12}$

Q.37. Evaluate: $\int_1^3 [|x-1| + |x-2| + |x-3|]\, dx$

Ans.

$$\text{Let } I = \int_1^3 [|x-1| + |x-2| + |x-3|]dx = \int_1^3 |x-1|dx + \int_1^3 |x-2|dx + \int_1^3 |x-3|dx$$

$$= \int_1^3 |x-1|dx + \int_1^2 |x-2|dx + \int_2^3 |x-2|dx + \int_1^3 |x-3|dx \quad \text{[By property of definite integral]}$$

$$= \int_1^3 |x-1|dx + \int_1^2 -(x-2)dx + \int_2^3 (x-2)dx + \int_1^3 -(x-3)dx \quad \left\{\begin{matrix} x-1 \geq 0, \text{ if } 1 \leq x \leq 3 \\ x-2 \leq 0, \text{ if } 1 \leq x \leq 2 \\ x-2 \geq 0, \text{ if } 2 \leq x \leq 3 \\ x-3 \leq 0, \text{ if } 1 \leq x \leq 3 \end{matrix}\right\}$$

$$= \left[\frac{(x-1)^2}{2}\right]_1^3 - \left[\frac{(x-2)^2}{2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^3 - \left[\frac{(x-3)^2}{2}\right]_1^3$$

$$= \left(\frac{4}{2} - 0\right) - \left(0 - \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right) - \left(-0 - \frac{4}{2}\right) = 2 + \frac{1}{2} + \frac{1}{2} + 2 = 5$$

Q.38. **Evaluate:** $\int_0^\pi \frac{x \tan x}{\sec x \,.\, \text{cosec } x} \text{dx}$

$$\text{Let } I = \int_0^\pi \frac{x \tan x}{\sec x \,.\, \text{cos ec } x} \text{dx} = \int_0^\pi \frac{x \,.\, \frac{\sin x}{\cos x}}{\frac{1}{\cos x} \cdot \frac{1}{\sin x}} \text{dx}$$

$$I = \int_0^\pi x \ \sin^2 \ x$$

$$= \int_0^\pi (\pi - x) \sin^2 (\pi - x) \text{dx} \qquad [\because \int_0^a f(x)\text{dx} = \int_0^a f(a-x)\text{dx}]$$

$$I = \int_0^\pi \pi \sin^2 x \,\text{dx} - \int_0^\pi x \sin^2 x \,\text{dx} \quad \Rightarrow \quad 2I = \frac{\pi}{2} \int_0^\pi 2\sin^2 \ x \ \text{dx}$$

$$= \frac{\pi}{2} \int_0^\pi (1 - \cos 2x)\text{dx} = \frac{\pi}{2} \int_0^\pi \text{dx} - \frac{\pi}{2} \int_0^\pi \cos 2x \,\text{dx} = \frac{\pi}{2} [x]_0^\pi - \frac{\pi}{2} \left[\frac{\sin 2x}{2}\right]_0^\pi$$

$$\Rightarrow 2I = \frac{\pi}{2}(\pi - 0) - \frac{\pi}{4}(\sin 2\pi - \sin 0)$$

$$\Rightarrow 2I = \frac{\pi^2}{2} - 0 \quad \Rightarrow \quad I = \frac{\pi^2}{2}$$

Q.39. Evaluate: $\int_0^{\pi/2} \log\,(\sin x)\text{dx}$

Ans.

Let $I = \int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - x\right)dx$ ...(i)

$I = \int_0^{\pi/2} \log\cos x\,dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} (\log\sin x + \log\cos x)dx = \int_0^{\pi/2} \log\sin x.\cos x\,dx$$

$$= \int_0^{\pi/2} \log\frac{2\sin x.\cos x}{2}dx = \int_0^{\pi/2} (\log\sin 2x - \log 2)dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \log 2\int_0^{\pi/2} dx = I_1 - \log 2[x]_0^{\pi/2}$$

$\left[\because \text{Here } \log\sin\left(\frac{2\pi}{2} - t\right) = \log\sin t \text{ and } \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)\right]$

$2I = I_1 - \frac{\pi}{2}\log 2$ ...(iii)

where $I_1 = \int_0^{\pi/2} \log\sin 2x\,dx$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$, If $x = 0, t = 0; x = \frac{\pi}{2}, t = \pi$

$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \log\sin t\,dt$

$\Rightarrow \quad I_1 = \frac{1}{2}\int_0^{2.\pi/2} \log\sin t\,dt \quad \Rightarrow \quad I_1 = \frac{1}{2}\times 2\int_0^{\pi/2} \log\sin t\,dt$

$\Rightarrow \quad I_1 = \int_0^{\pi/2} \log\sin x\,dx$ $\quad [\because \int_a^b f(x)dx = \int_a^b f(t)dt]$

$\Rightarrow \quad I_1 = I$

Putting it in (iii), we get

$$2I = I - \frac{\pi}{2}\log 2 \quad \Rightarrow \quad I = -\frac{\pi}{2}\log 2$$

Q.40. Evaluate: $\int_0^1 \cot^{-1}(1 - x + x^2)\,dx$

Ans.

Let $I = \int_0^1 \cot^{-1}(1 - x + x^2)dx$

$$= \int_0^1 \tan^{-1} \frac{1}{1 - x + x^2} dx \qquad \left[\because \cot^{-1} x = \tan^{-1} \frac{1}{x}\right]$$

$$= \int_0^1 \tan^{-1} \frac{x + (1 - x)}{1 - x(1 - x)} dx$$

$$= \int_0^1 \{\tan^{-1} x + \tan^{-1}(1 - x)\} dx \qquad \left[\because \tan^{-1}(x + y) = \tan^{-1} \frac{x + y}{1 - xy}\right]$$

$$= \int_0^1 \tan^{-1} x \; dx + \int_0^1 \tan^{-1}(1 - x) dx$$

$$= \int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1}(1 - (1 - x)) dx \quad \left[\because \int_0^a f(x) dx = \int_0^a f(a - x) dx\right]$$

$$= \int_0^1 \tan^{-1} x \, dx + \int_0^1 \tan^{-1} x \, dx = 2\int_0^1 \tan^{-1} x \, dx = 2\int_0^1 1.\tan^{-1} x \; dx$$

$$= 2\left\{[\tan^{-1} x.x]_0^1 - \int_0^1 \frac{1}{1 + x^2} . x. dx\right\} \quad \Rightarrow \quad 2\frac{\pi}{4} - \int_0^1 \frac{2x \, dx}{1 + x^2} = \frac{\pi}{2} - [\log|1 + x^2|]_0^1$$

$$= \frac{\pi}{2} - [\log 2 - \log 1] = \frac{\pi}{2} - \log 2$$

Q.41. Evaluate: $\int_0^1 x^2 (1 - x)^n \, \mathrm{dx}$

Ans.

Let $I = \int_0^1 x^2 (1 - x)^n \, \mathrm{dx}$

$$\Rightarrow \quad I = \int_0^1 (1 - x)^2 [1 - (1 - x)]^n \mathrm{dx} \qquad \left[\because \int_0^a f(x)\mathrm{dx} = \int_0^a f(a - x)\mathrm{dx}\right]$$

$$= \int_0^1 (1 - 2x + x^2) x^n \mathrm{dx} = \int_0^1 (x^n - 2x^{n+1} + x^{n+2}) \mathrm{dx}$$

$$= \left[\frac{x^{n+1}}{n+1} - 2.\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{n+3}\right]_0^1 = \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$$

$$= \frac{(n+2)(n+3) - 2(n+1)(n+3) + (n+1)(n+2)}{(n+1)(n+2)(n+3)}$$

$$= \frac{n^2 + 5n + 6 - 2n^2 - 8n - 6 + n^2 + 3n + 2}{(n+1)(n+2)(n+3)} = \frac{2}{(n+1)(n+2)(n+3)}$$

Q.42. Evaluate: $\int_1^3 (2x^2 + 5x)\,\mathrm{dx}$ as a limit of a sum.

Ans.

Let $f(x) = 2x^2 + 5x$

Here $a = 1,\ b = 3$ $\quad\therefore\quad h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$

$\Rightarrow nh = 2$

Also, $n \to \infty \Leftrightarrow h \to 0$.

$$\therefore \int_a^b f(x)\mathrm{dx} = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f\{a + (n-1)h\}]$$

$$\therefore \int_1^3 (2x^2 + 5x)\mathrm{dx} = \lim_{h\to 0} h[f(1) + f(1+h) + \ldots + f\{1 + (n-1)h\}]$$

$$= \lim_{h\to 0} h[\{2\times 1^2 + 5\times 1\} + \{2(1+h)^2 + 5(1+h)\} + \ldots + \{2(1-(n-1)h)^2 + 5((1+(n-1)h\}]$$

$$= \lim_{h\to 0} h[(2+5) + \{2 + 4h + 2h^2 + 5 + 5h\} + \ldots + \{2 + 4(n-1)h + 2(n-1)^2h^2 + 5 + 5(n-1)h\}]$$

$$= \lim_{h\to 0} h[7 + \{7 + 9h + 2h^2\} + \ldots + \{7 + 9(n-1)h + 2(n-1)^2h^2\}]$$

$$= \lim_{h\to 0} h[7n + 9h\{1 + 2 + \ldots + (n-1)\} + 2h^2\{1^2 + 2^2 + \ldots + (n-1)^2\}]$$

$$= \lim_{h\to 0}\left[7\mathrm{nh} + 9h^2\frac{(n-1).n}{2} + 2h^3\frac{(n-1).n\,(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[7(\mathrm{nh}) + \frac{9(\mathrm{nh})^2.\left(1-\frac{1}{n}\right)}{2} + \frac{2(\mathrm{nh})^3.\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)}{6}\right]$$

$$= \lim_{n\to\infty}\left[14 + \frac{36\left(1-\frac{1}{n}\right)}{2} + \frac{16\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)}{6}\right] \quad [\because \mathrm{nh} = 2]$$

$$= \lim_{n\to\infty}\left[14 + 18\left(1-\frac{1}{n}\right) + \frac{8}{3}\left(1-\frac{1}{n}\right).\left(2-\frac{1}{n}\right)\right]$$

$$= 14 + 18 + \frac{8}{3}\times 1\times 2 = 32 + \frac{16}{3} = \frac{96+16}{3} = \frac{112}{3}$$

Q.43. Evaluate: $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}\mathrm{dx}$

Ans.

$$\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x} dx = \int_0^{\pi/2} \frac{x}{1+\cos x} dx + \int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx$$

$$= \int_0^{\pi/2} \frac{x}{2\cos^2 x/2} dx + \int_0^{\pi/2} \frac{2\sin x/2.\cos x/2}{2\cos^2 x/2} dx$$

$$= \frac{1}{2} \int_0^{\pi/2} x \sec^2 \frac{x}{2} dx + \int_0^{\pi/2} \tan \frac{x}{2} dx$$

$$= \frac{1}{2} \left[x.2\tan \frac{x}{2}\right]_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} 1.2\tan \frac{x}{2} dx + \int_0^{\pi/2} \tan \frac{x}{2} dx$$

$$= \frac{1}{2} \left[\frac{\pi}{2}.2\tan\frac{\pi}{4} - 0\right] = \frac{\pi}{2}$$

Q.44. **Evaluate :** $\int_{-1}^{2} |x^3 - x| \, dx$

Ans.

If $x^3 - x = 0$

$\Rightarrow x(x^2 - 1) = 0$

$\Rightarrow x = 0$ or $x^2 = 1$

$\Rightarrow x = 0$ or $x = \pm 1$

$\Rightarrow x = 0, -1, 1$

Hence $[-1, 2]$ is divided into three sub intervals $[-1, 0]$, $[0, 1]$ and $[1, 2]$ such that

$$x^3 - x \geq 0 \quad \text{on} \quad [1, 0]$$

$$x^3 - x \leq 0 \quad \text{on} \quad [0, 1]$$

and $x^3 - x \geq 0$ on $[1, 2]$

Now, $\int_{-1}^{2} |x^3 - x| dx = \int_{-1}^{0} |x^3 - x| dx + \int_{0}^{1} |x^3 - x| dx + \int_{1}^{2} |x^3 - x| dx$

$$= \int_{-1}^{0} (x^3 - x) dx + \int_{0}^{1} -(x^3 - x) dx + \int_{1}^{2} (x^3 - x) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{x}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= \left\{0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right\} - \left\{\left(\frac{1}{4} - \frac{1}{2}\right) - 0\right\} + \left\{(4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right\}$$

$$= -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

Q.45. **Evaluate :** $\int_{0}^{2\pi} \frac{1}{1+e^{\sin x}} dx$

Ans.

Let $I = \int_{0}^{2\pi} \frac{1}{1+e^{\sin x}} dx \quad ...(i)$

Applying property $\int_{0}^{a} f(x) dx = \int_{0}^{a} f(a - x) dx$, we get

$$I = \int_{0}^{2\pi} \frac{dx}{1+e^{\sin (2\pi - x)}} = \int_{0}^{2\pi} \frac{dx}{1+e^{-\sin x}} = \int_{0}^{2\pi} \frac{dx}{1+\frac{1}{e^{\sin x}}}$$

$$I = \int_{0}^{2\pi} \frac{e^{\sin x} dx}{e^{\sin x}+1} \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_{0}^{2\pi} \frac{dx}{1+e^{\sin x}} + \int_{0}^{2\pi} \frac{e^{\sin x}}{1+e^{\sin x}} = \int_{0}^{2\pi} \frac{1+e^{\sin x}}{1+e^{\sin x}} dx = \int_{0}^{2\pi} dx = [x]_{0}^{2\pi}$$

$\Rightarrow 2I = 2\pi \quad \Rightarrow \quad I = \pi.$

Q.46. Evaluate $\int_0^{\pi} e^{2x}.\sin\left(\frac{\pi}{4}+x\right)\mathrm{dx}.$

Ans.

We have $I = \int_0^{\pi} e^{2x}.\sin\left(\frac{\pi}{4}+x\right)\mathrm{dx}$

Integrating by part, we get.

$$I = \left[\sin\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\right]_0^{\pi} - \int_0^{\pi}\cos\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\mathrm{dx}$$

$$= \frac{1}{2}\left[\sin\frac{5\pi}{4}.e^{2\pi} - \sin\frac{\pi}{4}\right] - \frac{1}{2}\int_0^{\pi} e^{2x}.\cos\left(\frac{\pi}{4}+x\right)\mathrm{dx}$$

$$= \frac{1}{2}\left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left[\left\{\cos\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\right\}_0^{\pi} + \int_0^{\pi}\sin\left(\frac{\pi}{4}+x\right).\frac{e^{2x}}{2}\mathrm{dx}\right]$$

$$= -\frac{e^{2\pi}+1}{2\sqrt{2}} - \frac{1}{2}\left[\cos\frac{5x}{4}.\frac{2^{2\pi}}{2} - \frac{1}{2}\cos\frac{\pi}{4}\right] - \frac{1}{4}\int e^{2x}.\sin\left(\frac{\pi}{4}+x\right)\mathrm{dx}$$

$$I = -\frac{e^{2\pi}+1}{2\sqrt{2}} - \frac{1}{4}.e^{2\pi}.\cos\frac{5\pi}{4} + \frac{1}{4\sqrt{2}} - \frac{1}{4}I$$

$$\frac{5I}{4} = -\frac{e^{2\pi}+1}{2\sqrt{2}} + \frac{e^{2\pi}}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}$$

$$= -\frac{e^{2\pi}+1}{2\sqrt{2}} + \frac{e^{2\pi}+1}{4\sqrt{2}} \quad \Rightarrow \quad \frac{e^{2\pi}+1}{4\sqrt{2}}(-2+1)$$

$$\frac{5I}{4} = -\frac{e^{2\pi}+1}{4\sqrt{2}}$$

$$I = -\frac{e^{2\pi}+1}{5\sqrt{2}}$$

Q.47. Evaluate : $\int_{-2}^{2}\frac{x^2}{1+5^x}\mathrm{dx}$ f

Ans.

Let $I = \int_{-2}^{2} \frac{x^2}{1+5^x} dx$ ... (i)

$= \int_{-2}^{2} \frac{(2+(-2)-x)^2}{1+5^{(2+(-2)-x)}} dx \qquad \left[\int_a^b f(x) dx = \int f(a+b-x) dx\right]$

$= \int_{-2}^{2} \frac{(-x)^2}{1+5^{-x}} dx = \int_{-2}^{2} \frac{x^2}{1+\frac{1}{5^x}} dx$

$I = \int_{-2}^{2} \frac{5^x x^2}{1+5^x} dx \quad \ldots(ii)$

Adding (i) and (ii), we get

$2I = \int_{-2}^{2} \frac{(1+5^x)x^2}{1+5^x} dx$

$= \int_{-2}^{2} x^2 dx = \left[\frac{x^3}{3}\right]_{-2}^{2}$

$\Rightarrow \quad 2I = \frac{1}{3}[8 - (-8)]$

$\Rightarrow \quad I = \frac{16}{3\times2} = \frac{8}{3}$

Q.48. **Find :** $\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

Ans.

Let $I = \int [\log(\log x) + \frac{1}{(\log x)^2}] \, dx$

Let $\log x = t \quad \Rightarrow \quad x = e^t \quad \Rightarrow \quad dx = e^t \, dt$

$\therefore I \quad = \int \left\{\log t + \frac{1}{t^2}\right\} e^t \, dt$

$= \int \left\{\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right\} e^t \, dt$

$= \int \left(\log t + \frac{1}{t}\right) e^t + \left(-\frac{1}{t} + \frac{1}{t^2}\right) e^t$

$= e^t . \log t - \frac{1}{t} . e^t + C \qquad [\because \int (f(x) + f'(x)) e^x dx = f(x) e^x + C]$

Put $t = \log x$

$e^{\log x} \log(\log x) - \frac{1}{\log x} e^{\log x} + C$

$\Rightarrow \; x . \log(\log x) - \frac{x}{\log x} + C$

Long Answer Questions-I (OIQ)

[4 Mark]

Q.1. Evaluate : $\int \frac{\tan\theta + \tan^3\theta}{1 + \tan^3\theta} d\theta$

Ans.

Let $I = \int \frac{\tan\theta + \tan^3\theta}{1 + \tan^3\theta} d\theta = \int \frac{\tan\theta(1 + \tan^2\theta)}{1 + \tan^3\theta} d\theta = \int \frac{\tan\theta \sec^2\theta}{1 + \tan^3\theta} d\theta$

Let $\tan\theta = z \quad \Rightarrow \quad \sec^2\theta \, d\theta = dz$

$\therefore I = \int \frac{z dz}{1 + z^3} = \int \frac{z dz}{(1 + z)(z^2 - z + 1)}$

Now $\frac{z}{(1+z)(z^2-z+1)} = \frac{A}{1+z} + \frac{Bz+C}{z^2-z+1} \qquad \Rightarrow \qquad z = A(z^2 - z + 1) + (Bz + C)(1 + z)$

Putting $z = -1$, we get, $A = -\frac{1}{3}$

Putting $z = 0$, we get, $C = \frac{1}{3}$

Putting $z = 1$, we get, $B = \frac{1}{3}$

$\therefore \frac{z}{(1+z)(z^2-z+1)} = \frac{-1}{3(1+z)} + \frac{\frac{1}{3}z+\frac{1}{3}}{z^2-z+1}$

$$\Rightarrow I = -\frac{1}{3}\int \frac{dz}{1+z} + \frac{1}{3}\int \frac{z+1}{z^2-z+1}dz = -\frac{1}{3}\log\left|1+z\right| + \frac{1}{3\times 2}\int \frac{2z-1+3}{z^2-z+1}dz$$

$$= -\frac{1}{3}\log\left|1+z\right| + \frac{1}{6}\int \frac{2z-1}{z^2-z+1}dz + \frac{1}{2}\int \frac{dz}{z^2-z+1}$$

$$I = -\frac{1}{3}\log\left|1+z\right| + \frac{1}{6}\log\left|z^2-z+1\right| + I_1 + C \quad ...(i)$$

Where

$$I_1 = \frac{1}{2}\int \frac{dz}{z^2-z+1} = \frac{1}{2}\int \frac{dz}{\left(z-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}.\frac{2}{\sqrt{3}} + \tan^{-1}\left(\frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$\therefore\ I = -\frac{1}{3}\log\left|1+z\right| + \frac{1}{6}\log\left|z^2-z+1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2z-1}{\sqrt{3}}\right) + C$$

Putting $z = \tan\theta$

$$\therefore\ I = -\frac{1}{3}\log\left|1+\tan\theta\right| + \frac{1}{6}\log\left|\tan^2\theta - \tan\theta + 1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$$

Evaluate: $\int e^x \frac{x^2+1}{(x+1)^2}dx$

Q.2.

Ans.

Let $I = \int e^x \frac{x^2+1}{(x+1)^2} dx = \int e^x \left(1 - \frac{2x}{(x+1)^2}\right) dx = \int e^x - 2\left(\frac{e^x x.\, dx}{(x+1)^2}\right)$

$= e^x - 2\int e^x \frac{x+1-1}{(x+1)^2} dx = e^x - 2\int e^x \left[\frac{1}{x+1} + \frac{-1}{(x+1)^2}\right] dx$

$= e^x - 2e^x . \frac{1}{x+1} + C$ [Note: $\int e^x \{f(x) + f'(x)\} dx = e^x . f(x) + C$]

Evaluate: $\int_0^{\pi/2} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta} d\theta$

Q.3.

Ans.

Let $t = \sin^2$ q, then dt = 2sin q cos q dq = sin 2q . dq

and $\sin^4$ q + $\cos^4$ q = $t^2 + (1 - t)^2$

$= 2t^2 - 2t + 1 = 2\left(t^2 - t + \frac{1}{2}\right) = 2\left[\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]$

Now, limits are, when θ = 0, $t = 0$ and when $\theta = \frac{\pi}{2}, t = 1$

Therefore $\int_0^{\pi/2} \frac{\sin 2\theta .\, d\theta}{\sin^4\theta + \cos^4\theta} = \int_0^1 \frac{dt}{2\left[\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]}$

$= \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} \left[\tan^{-1} \frac{\left(t - \frac{1}{2}\right)}{\frac{1}{2}}\right]_0^1 = [\tan^{-1}(2t-1)]_2^t$

$= \tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

Q.4. Evaluate: $\int_0^{\pi/2} \sin 2x \log \tan x \, dx$

Ans.

Let $I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx$

$$\therefore \quad I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx \qquad \left[\text{since } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/2} \sin(\pi - 2x) \log \cot x \, dx = \int_0^{\pi/2} \sin 2x \log (\tan x)^{-1} dx$$

$$= -\int_0^{\pi/2} \sin 2x \log \tan x \, dx = -I$$

$$\therefore \quad 2I = 0 \quad \Rightarrow \quad I = 0$$

Q.5. Find: $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}} dx$

Ans.

Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}} dx$

Putting $x + 1 = t^2 \Rightarrow dx = 2t \, dt$

$$I = 2\int \frac{t^2+1}{t^4+t^2+1} dt = 2\int \frac{1+\left(\frac{1}{t}\right)^2}{t^2+\frac{1}{t^2}+1} dt \quad [\text{Dividing } N^r \text{ and } D^r \text{ by } t^2]$$

$$= 2\int \frac{1 + \left(\frac{1}{t}\right)^2}{\left(t - \frac{1}{t}\right)^2 + 3} dt$$

Now, Put $t - \frac{1}{t} = z \quad \Rightarrow \quad \left(1 + \frac{1}{t^2}\right) dt = dz$, we get

$$I = 2\int \frac{dz}{z^2 + 3} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{z}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{t - \frac{1}{t}}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{t^2 - 1}{\sqrt{3}t}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3(x+1)}}\right) + C$$

Q.6. **Find:** $\int \frac{1}{\sqrt{\sin^3 x \sin (x + \alpha)}} dx;\ \alpha \neq n\pi,\ n \in Z$

Ans.

Let $I = \int \frac{1}{\sqrt{\sin^3 x \sin (x+\alpha)}} dx$

$\sin^3 x \sin (x + a) = \sin^3 x (\sin x \,.\, \cos a + \cos x \,.\, \sin a)$

$= \sin^4 x (\cos a + \cot x \,.\, \sin a)$

$$\therefore\ I = \int \frac{1}{\sqrt{\sin^3 x \sin (x + \alpha)}} . dx = \int \frac{1}{\sin^2 x \sqrt{\cos \alpha + \cot x \sin \alpha}} . dx$$

Putting $\cos a + \cot x \,.\, \sin a = t$ so that $-\operatorname{cosec}^2 x \,.\, \sin a \,.\, dx = dt$

$$\therefore\ I = \int -\frac{1}{\sin \alpha \sqrt{t}} . dt = -\frac{1}{\sin \alpha} \int t^{-1/2} dt = -\frac{1}{\sin \alpha} \left(\frac{t^{1/2}}{1/2}\right) + C$$

$$= -2\operatorname{cosec} \alpha.\ \sqrt{t} + C = -2 \operatorname{cosec} \alpha (\cos \alpha + \cot x \sin \alpha)^{1/2} + C$$

Q.7. Find: $-\int 5^{5^{5^x}} . 5^{5^x} . 5^x \, dx$

Ans.

Let $I = \int 5^{5^{5^x}} . 5^{5^x} . 5^x \, dx$

Putting $5^x = t \Rightarrow 5^x . \log 5 \, dx = dt$ or $5^x . dx = \frac{dt}{(\log 5)}$

Therefore, $I = \int 5^{5^{5^x}} . 5^{5^x} . 5^x . dx = \int 5^{5^t} . 5^t . \frac{dt}{(\log 5)} = \frac{1}{(\log 5)} \int 5^{5^t} . 5^t . dt$

Again, putting $5^t = u, \quad 5^t dt = \frac{du}{(\log 5)}$

Therefore, $I = \frac{1}{(\log 5)} \int 5^u . \frac{du}{(\log 5)}$

$= \frac{1}{(\log 5)^2} \int 5^u \mathrm{du} = \frac{5^u}{(\log 5)^2 . (\log 5)} + C$

$= \frac{5^u}{(\log 5)^3} + C = \frac{5^{5^t}}{(\log 5)^3} + C = \frac{5^{5^{5^x}}}{(\log 5)^3} + C$

Q.8. Show that: $\int_0^{\pi/2} \log(\tan\theta + \cot\theta) \, d\theta = \pi \log 2$

Ans.

Let $I = \int_0^{\pi/2} \log (\tan \theta + \cot \theta)\, d\theta$

$$= \int_0^{\pi/2} \log \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}\right) d\theta \quad \Rightarrow \quad \int_0^{\pi/2} \log \left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta \cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log \left(\frac{1}{\cos\theta \ \sin\theta}\right) d\theta$$

$$= -\int_0^{\pi/2} \log (\cos \theta \sin \theta) d\theta \quad \left[\because \log \frac{1}{m} = \log m^{-1} = -\log m\right]$$

$$= -\int_0^{\pi/2} (\log \cos\theta + \log \sin \theta) d\theta \quad \Rightarrow \quad -\int_0^{\pi/2} \log \cos \theta\, d\theta - \int_0^{\pi/2} \log \sin \theta\, d\theta$$

$$= -\int_0^{\pi/2} \log \cos \left(\frac{\pi}{2} - \theta\right) d\theta - \int_0^{\pi/2} \log \sin \theta \, d\theta$$

$$= -\int_0^{\pi/2} \log \sin \theta \, d\theta - \int_0^{\pi/2} \log \sin \theta \, d\theta$$

$$= -2\int_0^{\pi/2} \log \sin\theta d\theta = -2\left(-\frac{\pi}{2} \log 2\right) = \pi \log 2$$

$$\left[\because \int_0^{\pi/2} \log \sin x dx = \frac{-\pi}{2} \log 2\right]$$

[**Note:** If this question comes in exam, students are advised to give complete solution of $\int_0^{\pi/2} \log \sin \theta\, d\theta$.]

Long Answer Questions-II (PYQ)

[6 Mark]

Q.1. Evaluate: $\int(\sqrt{\cot x} + \sqrt{\tan x})\,dx$

Ans.

Let $I = \int(\sqrt{\cot\ x} + \sqrt{\tan\ x})\,dx$

$I = \int\left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)dx = \int \frac{(\cos x + \sin x)}{\sqrt{\sin x.\cos x}}dx$

Let $(\sin\ x - \cos\ x) = t \Rightarrow (\cos\ x + \sin\ x)\ dx = dt$

Also $\sin\ x - \cos\ x = t$

$\Rightarrow(\sin\ x - \cos\ x)^2 = t^2 \Rightarrow \sin^2\ x + \cos^2\ x - 2\ \sin\ x\ .\ \cos\ x = t^2$

$\Rightarrow 1 - 2\ \sin\ x\ .\ \cos\ x = t^2 \Rightarrow$

Therefore, $I = \int \frac{dt}{\sqrt{\frac{1-t^2}{2}}} = \sqrt{2}\int\frac{dt}{\sqrt{1 = t^2}}$

$= \sqrt{2}\sin^{-1}t + C = \sqrt{2}\sin^{-1}(\sin x - \cos x) + C$

Q.2. Evaluate: $\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}dx$

Ans.

$$\text{Let } I = \int \frac{1}{\sin^4 x + \sin^2 x.\ \cos^2 x + \cos^4 x} dx$$

Dividing N^r and D^r by $\cos^4 x$, we get

$$\text{Put } I = \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx \quad z = \tan x \Rightarrow dz = \sec^2 x\, dx$$

$$\therefore \quad I = \int \frac{(1+z^2)dz}{z^4+z^2+1} = \int \frac{z^2\left(1+\frac{1}{z^2}\right)}{z^2\left\{z^2+\frac{1}{z^2}+1\right\}} dz$$

$$= \int \frac{\left(1+\frac{1}{z^2}\right)}{\left(z-\frac{1}{z}\right)^2+3} dz = \int \frac{\left(1+\frac{1}{z^2}\right) dz}{\left(z-\frac{1}{z}\right)^2+(\sqrt{3})^2}$$

Again, let $z - \frac{1}{z} = t \quad \Rightarrow \quad \left(1+\frac{1}{z^2}\right) dz = dt$

$$\Rightarrow \; I = \int \frac{dt}{t^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{t}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{3}}\right) + C \quad \left[\because z - \frac{1}{z} = t\right]$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z^2-1}{\sqrt{3}z}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{3}\tan x}\right) + C$$

Q.3. **Evaluate:** $\int \frac{1}{\cos^4 x + \sin^4 x} dx$

Ans.

$$\text{Let } I = \int \frac{1}{\cos^4 x + \sin^4 x} dx$$

$$= \int \frac{\sec^4 x\, dx}{1+\tan^4 x} \quad [\text{Dividing N}^r \text{ and D}^r \text{ by } \cos^4 x]$$

$$= \int \frac{\sec^2 x.\ \sec^2 x\, dx}{1+\tan^4 x} = \int \left(\frac{1+\tan^2 x}{1+\tan^4 x}\right) \sec^2 x dx$$

Put tan x = z ⇒ sec2 x dx = dz

$$\therefore \; I = \int \left(\frac{1+z^2}{1+z^4}\right) dz = \int \frac{\left(\frac{1}{z^2}+1\right)}{\left(\frac{1}{z^2}+z^2\right)} dz \quad \text{[Dividing Nr and Dr by z2]}$$

$$= \int \frac{\left(1+\frac{1}{z^2}\right) dz}{\left(z-\frac{1}{z}\right)^2+2}$$

Let $z - \frac{1}{z} = t \quad \Rightarrow \quad \left(1+\frac{1}{z^2}\right)\mathrm{dz} = \mathrm{dt}$

$$\therefore I = \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2}}\right) + C \quad \left[\text{Putting } t = z - \frac{1}{z}\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z^2-1}{\sqrt{2}z}\right) + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + C$$

Q.4. **Evaluate:** $\int \frac{2\mathrm{x}^2+1}{\mathrm{x}^2(\mathrm{x}^2+4)}\mathrm{dx}.$

Ans.

Let $I = \int \frac{2x^2+1}{x^2(x^2+4)} dx$

Put $x^2 = y$, we get

$\therefore \quad \frac{2x^2+1}{x^2(x^2+4)} = \frac{2y+1}{y(y+4)}$

Now, $\frac{2y+1}{y(y+4)} = \frac{A}{y} + \frac{B}{y+4}$

$\Rightarrow 2y + 1 = A(y+4) + By \quad \Rightarrow 2y + 1 = (A+B)y + 4A$

$\Rightarrow 4A = 1 \quad \Rightarrow$

and $A + B = 2 \quad \Rightarrow B =$

$$\therefore \quad I = \int \frac{1}{4x^2} dx + \int \frac{7dx}{4(x^2+4)}$$

$$= \frac{1}{4}\left(\frac{x^{-2+1}}{-2+1}\right) + \frac{7}{4} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + C = -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\frac{x}{2} + C$$

Q.5. Evaluate: $\int \frac{x^4 dx}{(x-1)(x^2+1)}$

Ans.

Let $I = \int \frac{x^4 dx}{(x-1)(x^2+1)}$

Suppose $\frac{x^4}{(x-1)(x^2+1)} = \frac{x^4-1+1}{(x-1)(x^2+1)} = x+1+\frac{1}{(x-1)(x^2+1)}$

Also, let $\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$\Rightarrow \quad 1 = A(x^2+1) + (Bx+C)(x-1)$

Equating coefficients of similar terms, we get

$A+B=0$
$-B+C=0$
$A-C=1 \Rightarrow B=C$
$A-B=1$
$A+B=0$

$\Rightarrow \quad 2A=1 \quad \Rightarrow \quad A=\frac{1}{2} \quad \Rightarrow \quad B=-\frac{1}{2}=C$

$$\therefore \; I = \int \left(x+1+\frac{\frac{1}{2}}{x-1} - \frac{1}{2}\frac{x+1}{x^2+1}\right) dx$$

$$= \frac{x^2}{2} + x + \frac{1}{2}\log\left|x-1\right| - \frac{1}{4}\int \frac{2x}{x^2+1}dx - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{x^2}{2} + x + \frac{1}{2}\log\left|x-1\right| - \frac{1}{4}\log\left|x^2+1\right| - \frac{1}{2}\tan^{-1} x + C$$

Q.6. Evaluate: $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

Ans.

Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin\, x + \cos\, x} dx$ ..(i)

$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\ \left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$ $\left[\because\ \int\limits_0^a f(x)dx = \int\limits_0^a f(a-x)dx\right]$

$I = \int\limits_0^{\pi/2} \frac{\cos^2 x}{\cos\, x + \sin\, x} dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \int\limits_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin\, x + \cos\, x} dx = \int\limits_0^{\pi/2} \frac{dx}{\sin\, x + \cos\, x}$$

$$= \int\limits_0^{\pi/2} \frac{dx}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\, x + \frac{1}{\sqrt{2}}\sin\, x\right)} = \frac{1}{\sqrt{2}} \int\limits_0^{\pi/2} \frac{dx}{\cos\, x.\ \cos\frac{\pi}{4} + \sin\, x.\ \sin\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} \int\limits_0^{\pi/2} \frac{dx}{\cos\ \left(x - \frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}} \int\limits_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right) dx$$

$[\therefore\ \cos\ (A - B) = \cos A \cos B + \sin A \sin B]$

$= \frac{1}{\sqrt{2}}\left[\log\left\{\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right\}\right]_0^{\pi/2}$ $\left[\because\ \int \sec\, x dx = \log(\sec\, x + \tan\, x)\right]$

$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \log\left\{\sec\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{4}\right)\right\}\right]$

$= \frac{1}{\sqrt{2}}\left[\log\ (\sqrt{2} + 1) - \log\left(\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right)\right]$

$= \frac{1}{\sqrt{2}}\left[\log\ (\sqrt{2} + 1) - \log\ (\sqrt{2} - 1)\right] = \frac{1}{\sqrt{2}}\log\left\{\frac{\sqrt{2}+1}{\sqrt{2}-1}\right\}$

$= \frac{1}{\sqrt{2}}\log\left\{\frac{(\sqrt{2}+1)^2}{2-1}\right\} = \frac{1}{\sqrt{2}}\log\ (\sqrt{2} + 1)^2 = \frac{2}{\sqrt{2}}\log\ (\sqrt{2} + 1)$

Hence, $I = \frac{1}{\sqrt{2}}\log\ (\sqrt{2} + 1)$

$$\text{Evaluate: } \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Q.7.

Ans.

$$\text{Let } I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) . \sin\left(\frac{\pi}{2} - x\right) . \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx \qquad \left[\begin{array}{l}\text{By Property} \\ \int_0^a f(x)dx = \int_0^a f(a-x)dx\end{array}\right]$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x . \sin x}{\cos^4 x + \sin^4 x} dx \qquad \left[\because \sin\left(\frac{\pi}{2} - x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2} - x\right) = \sin x\right]$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x . \sin x}{\sin^4 x + \cos^4 x} dx - \int_0^{\pi/2} \frac{x \sin x . \cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x . \sin x}{\sin^4 x + \cos^4 x} - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x . \cos x \, dx}{\sin^4 x + \cos^4 x} = \frac{\pi}{2} \int_0^{\pi/2} \frac{\frac{\sin x . \cos x}{\cos^4 x} dx}{\tan^4 x + 1}$$

[Dividing numerator and denominator by $\cos^4 x$]

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x . \cos x \, dx}{\sin^4 x + \cos^4 x} = \frac{\pi}{2} \int_0^{\pi/2} \frac{\frac{\sin x . \cos x}{\cos^4 x} dx}{\tan^4 x + 1}$$

Let $\tan^2 x = z$; $2\tan x . \sec^2 x \, dx = dz$

The limits are, when $x = 0$, $z = 0$;

$$\therefore \quad 2I = \frac{\pi}{4} \int_0^{\infty} \frac{dz}{1 + z^2} = \frac{\pi}{4} [\tan^{-1} z]_0^{\infty} = \frac{\pi}{4} (\tan^{-1}\infty - \tan^{-1} 0)$$

$$\therefore 2I = \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right) \qquad \Rightarrow \qquad I = \frac{\pi^2}{16}$$

Q.8. Evaluate: $\int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)dx$

Ans.

Let $I = 2\int_0^{\pi/2} \sin x.\ \cos x.\ \tan^{-1}(\sin x)\,dx$

Put $\sin x = z \Rightarrow \cos x\, dx = dz$

The limits are, when $x = 0, z = \sin 0 = 0;\ x = \frac{\pi}{2}, z = \sin\frac{\pi}{2} = 1$

$\therefore I = 2\int_0^1 z\tan^{-1}(z)dz = 2\left[\tan^{-1}z.\frac{z^2}{2}\right]_0^1 - 2\int_0^1 \frac{1}{1+z^2}\cdot\frac{z^2}{2}dz$

$= 2\left[\frac{\pi}{4}.\frac{1}{2} - 0\right] - \frac{2}{2}\int_0^1 \frac{z^2}{1+z^2}dz$

$= \frac{\pi}{4} - \int_0^1 \frac{1+z^2-1}{1+z^2}dz = \frac{\pi}{4} - \int_0^1 dz + \int_0^1 \frac{dz}{1+z^2}$

$= \frac{\pi}{4} - [z]_0^1 + [\tan^{-1}z]_0^1 = \frac{\pi}{4} - 1 + \left[\frac{\pi}{4} - 0\right] = \frac{\pi}{2} - 1$

Q.9. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

Ans.

$$\text{Let}\quad I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\cot\ x}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\frac{\sqrt{\cos\ x}}{\sqrt{\sin\ x}}}$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin\ x}}{\sqrt{\sin\ x}+\sqrt{\cos\ x}}dx \quad \ldots(i)$$

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin\ \left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\ \left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx \quad \left[\because \int_a^b f(x)\,dx=\int_a^b f(a+b-x)dx\right]$$

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\ x}}{\sqrt{\sin\ x}+\sqrt{\cos\ x}}dx \qquad \ldots(ii)$$

Adding (i) and (ii), we get

$$2I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin\ x}+\sqrt{\cos\ x}}{\sqrt{\sin\ x}+\sqrt{\cos\ x}}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx$$

$$\Rightarrow\ 2I=[x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{\pi}{3}-\frac{\pi}{6}=\frac{2\pi-\pi}{6}=\frac{\pi}{6} \quad \Rightarrow \quad I=\frac{\pi}{12}$$

Q.10. Evaluate: $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$

Ans.

$$\text{Let } I=\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx \quad ...(i)$$

$$\Rightarrow I=\int_0^{\pi} \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}}\,dx \qquad \left(\because \int_0^a f(x)dx=\int_0^a f(a-x)\,dx\right)$$

$$I=\int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}\,dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi} \frac{e^{\cos x}+e^{-\cos x}}{e^{\cos x}+e^{-\cos x}}\,\mathrm{dx}=\int_0^{\pi}\mathrm{dx}=[x]_0^{\pi}=\pi \quad \Rightarrow \quad I=\frac{\pi}{2}$$

Q.11. Evaluate: $\int_0^{\pi} \frac{\mathrm{x\,dx}}{\mathrm{a^2cos^2x+b^2sin^2x}}$

Ans.

$$\text{Let } I=\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x} \quad ...(i)$$

$$I=\int_0^{\pi} \frac{\pi-x}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)} \qquad \left[\text{By using } \int_0^a f(x)\,dx=\int_0^a f(a-x)dx\right]$$

$$I=\int_0^{\pi} \frac{\pi-x}{a^2\cos^2 x+b^2\sin^2 x} \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{\pi}{a^2\cos^2 x+b^2\sin^2 x}dx \quad \Rightarrow \quad I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$$

Divide numerator and denominator by $\cos^2 x$, we get

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{\sec^2 x dx}{a^2+b^2\tan^2 x}$$

$$I=\pi\int_0^{\frac{\pi}{2}}\frac{\sec^2 x dx}{a^2+b^2\tan^2 x} \qquad \left[\text{By using }\int_0^{2a}f(x)dx=2\int_0^{a}f(x)dx\right]$$

Put $b\tan x=t \Rightarrow b\sec^2 x\, dx=dt$

The limits are, when $x=0$, $t=0$ and $x=\frac{\pi}{2}, t=\infty$,

$$I=\frac{\pi}{b}\int_0^{\infty}\frac{dt}{a^2+t^2}=\frac{\pi}{b}.\frac{1}{a}\tan^{-1}\frac{t}{a}\Big]_0^{\infty}$$

$$I=\frac{\pi}{ab}(\tan^{-1}\infty-\tan^{-1}0)=\frac{\pi}{ab}.\frac{\pi}{2} \qquad \Rightarrow \qquad I=\frac{\pi^2}{2ab}$$

Q.12. **Evaluate:** $\int_1^4 [|x-1|+|x-2|+|x-4|]\,dx$

Ans.

Given, $I = \int_{1}^{4} [|x-1| + |x-2| + |x-4|] dx$

$$= \int_{1}^{4} (x-1)\,dx + \int_{1}^{2} -(x-2)\,dx + \int_{2}^{4} (x-2)dx + \int_{1}^{4} -(x-4)\,dx$$

$$= \left[\frac{x^2}{2} - x\right]_1^4 + \left[-\frac{x^2}{2} + 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 + \left[-\frac{x^2}{2} + 4x\right]_1^4$$

$$= \left(\frac{16}{2} - 4 - \frac{1}{2} + 1\right) + \left(-2 + 4 + \frac{1}{2} - 2\right) + \left(\frac{16}{2} - 8 - 2 + 4\right) + \left(-\frac{16}{2} + 16 + \frac{1}{2} - 4\right)$$

$$= \left(5 - \frac{1}{2}\right) + \frac{1}{2} + 2 + 4 + \frac{1}{2} = 11 + \frac{1}{2} = \frac{23}{2}$$

Q.13. **Evaluate $\int_1^3 (3x^2 + 2x)\,dx$ as limit of sums.**

Ans.

$\int_1^3 (3x^2 + 2x)dx$

We have to solve this by the help of limit of sum.

So, $a = 1, b = 3, h = \frac{3-1}{n} \Rightarrow nh = 2$

$\because \quad \int_1^3 (3x^2 + 2x)dx = \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \dots f(1 + (\overline{n-1})h]$ ...(*i*)

$f(1) = 3(1)^2 + 2(1)$

$f(1+h) = 3\,(1+h)^2 + 2(1+h) = 3h^2 + 8h + 5$

$f(1+2h) = 3\,(1+2h)^2 + 2(1+2h) = 12h^2 + 16h + 5$

⋮ ⋮ ⋮ ⋮ ⋮

⋮ ⋮ ⋮ ⋮ ⋮

$f(1 - (\overline{n-1})h) = 3\,(1 + (n-1)\,h)^2 + 2(1 + (n-1)h)$

$= 3\,(n-1)^2\,h^2 + 8\,(n-1)h + 5$

By putting all values in equation (*i*), we get

$$\int_1^3 (3x^2 + 2x)dx = \lim_{h \to 0} h[(5) + (3h^2 + 8h + 5) + (12h^2 + 16h + 5) + \dots$$

$$+ [3(n-1)^2\,h^2 + 8(n-1)h + 5]$$

$$= \lim_{h \to 0} h[3h^2\{1 + 4 + \dots + (n-1)^2\} + 8h\{1 + 2 + \dots + (n-1)\} + 5n]$$

$$[\because \{1+4+....+(n-1)^2 = \frac{(n-1)(2n-1)n}{6} \text{ and } \{1+2+...+(n-1) = \frac{(n-1)n}{2}]$$

$$= \lim_{h\to 0}\left[\frac{(nh-h)(nh)(2nh-h)}{2}+4(nh-h)(nh)+5nh\right]$$

$$= \lim_{h\to 0}\left[\frac{(2-h)(2)(4-h)}{2}+4(2-h)(2)+10\right]$$

$$= \left[\frac{2\times 2\times 4}{2}+4\times 2\times 2+10\right]=34 \quad \text{[by applying limit]}$$

Q.14. Evaluate the following: $\int_0^{3/2} |x\cos\pi x|dx$

Ans.

$\int_0^{3/2} |x\cos\pi x|\, dx$

As we know, $\cos x = 0 \quad \Rightarrow \quad x = (2n-1)\frac{\pi}{2}, n\in Z$

$\therefore \quad \cos\pi x = 0 \quad \Rightarrow \quad x = \frac{1}{2}, \frac{3}{2}$

For $0 < x < \frac{1}{2}, \quad x > 0$

$\cos\pi x > 0 \quad \Rightarrow \quad x\cos\pi x > 0$

For $\frac{1}{2} < x < \frac{3}{2}, \quad x > 0$

$\cos\pi x < 0 \quad \Rightarrow \quad x\cos\pi x < 0$

$\therefore \quad \int_0^{3/2} |x\cos\pi x|\, dx$

$$= \underset{\text{I}}{\int_0^{1/2} x\cos\pi x\, dx} + \underset{\text{II}}{\int_{1/2}^{3/2} (-x\cos\pi x)dx} \quad ...(i)$$

$$= \left[x\frac{\sin\pi x}{\pi}\right]_0^{1/2} - \int_0^{1/2} 1.\frac{\sin\pi x}{\pi}dx - \left[\frac{x\sin\pi x}{\pi}\right]_{1/2}^{3/2} - \int_{1/2}^{3/2}\frac{\sin\pi x}{\pi}dx$$

$$= \left[\frac{x}{\pi}\sin\pi x + \frac{1}{\pi^2}\cos\pi x\right]_0^{1/2} - \left[\frac{x}{\pi}\sin\pi x + \frac{1}{\pi^2}\cos\pi x\right]_{1/2}^{3/2}$$

$$= \left(\frac{1}{2\pi}+0-\frac{1}{\pi^2}\right)-\left(-\frac{3}{2\pi}-\frac{1}{2\pi}\right) = \frac{5}{2\pi}-\frac{1}{\pi^2}$$

Q.15. Evaluate: $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Ans.

Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ ...(*i*)

$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\Rightarrow$ $I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Put $\cos x = z$ $\Rightarrow$ $-\sin x\, dx = dz$

The limits are, when $x = 0 \Rightarrow z = 1$ and $x = \pi \Rightarrow z = -1$

$\therefore$ $2I = -\pi \int_1^{-1} \frac{dz}{1+z^2} = \pi \int_{-1}^{1} \frac{dz}{1+z^2} = \pi[\tan^{-1} z]_{-1}^{1}$

$= \pi[\tan^{-1} 1 - \tan^{-1}(-1)] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$

Q.16. Evaluate: $\int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x} dx$

Ans.

Let $I=\int_0^{\pi/2}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx=\int_0^{\pi/2}\frac{\cos^2 x}{\cos^2 x+4(1-\cos^2 x)}dx$

$$=\int_0^{\pi/2}\frac{\cos^2 x}{4-3\cos^2 x}dx=-\frac{1}{3}\int_0^{\pi/2}\frac{4-3\cos^2 x-4}{4-3\cos^2 x}dx$$

$$=-\frac{1}{3}\int_0^{\pi/2}\left(1-\frac{4}{4-3\cos^2 x}\right)dx=-\frac{1}{3}\int_0^{\pi/2}dx+\frac{4}{3}\int_0^{\pi/2}\frac{dx}{4-3\cos^2 x}=\frac{-1}{3}[x]_0^{\pi/2}+\frac{4}{3}\int_0^{\pi/2}\frac{\sec^2 x\,dx}{4\sec^2 x-3}$$

$$=-\frac{1}{3}.\frac{\pi}{2}+\frac{4}{3}\int_0^{\pi/2}\frac{\sec^2 x dx}{4(1+\tan^2 x)-3}$$

$$\left[\begin{array}{l}\text{Let}\tan x=z\ \Rightarrow\ \sec^2 x\,dx=dz\\ \text{The limits are, } x=\frac{\pi}{2}\Rightarrow z=\infty;\ x=0\Rightarrow z=0\end{array}\right]$$

$$=-\frac{\pi}{6}+\frac{4}{3}\int_0^{\infty}\frac{dz}{4+4z^2-3}=-\frac{\pi}{6}+\frac{4}{3\times 4}\int_0^{\infty}\frac{dz}{z^2+\left(\frac{1}{2}\right)^2}$$

$$=-\frac{\pi}{6}+\frac{1}{3}2.\left[\tan^{-1}\frac{z}{1/2}\right]_0^{\infty}=-\frac{\pi}{6}+\frac{2}{3}[\tan^{-1}2z]_0^{\infty}$$

$$=-\frac{\pi}{6}+\frac{2}{3}[\tan^{-1}\infty-\tan^{-1}0]=-\frac{\pi}{6}+\frac{2}{3}\left[\frac{\pi}{2}-0\right]=-\frac{\pi}{6}+\frac{\pi}{3}=\frac{\pi}{6}$$

Q.17. Evaluate: $\int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx$

Ans.

Let $$I=\int_0^{\pi}\frac{x}{1+\sin\alpha\sin x}dx=\int_0^{\pi}\frac{\pi-x}{1+\sin\alpha.\sin(\pi-x)}dx$$

$$=\int_0^{\pi}\frac{\pi}{1+\sin\alpha\sin x}dx-\int_0^{\pi}\frac{x}{1+\sin\alpha.\sin x}dx$$

$$I=\pi\int_0^{\pi}\frac{dx}{1+\sin\alpha.\sin x}-I$$

$\Rightarrow$ $$2I=\pi\int_0^{\pi}\frac{dx}{1+\sin\alpha.\sin x}=\pi\int_0^{\pi}\frac{dx}{1+\sin\alpha.\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$=\pi\int_0^{\pi}\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{1+\tan^2\frac{x}{2}+2\sin\alpha.\tan\frac{x}{2}}=\pi\int_0^{\pi}\frac{\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2}+2\sin\alpha.\tan\frac{x}{2}+1}$$

Let $\tan\frac{x}{2}=t\ \Rightarrow\ \sec^2\frac{x}{2}dx=2dt;\ x=0\ \Rightarrow\ t=0$ and $x=\pi\Rightarrow\ t=\infty$

$\therefore$ $$2I=2\pi\int_0^{\infty}\frac{dt}{t^2+2\sin\alpha\, t+1}$$

$$I=\pi\int_0^{\infty}\frac{dt}{t^2+2\sin\alpha\, t+\sin^2\alpha-\sin^2\alpha+1}$$

$$=\pi\int_0^{\infty}\frac{dt}{(t+\sin\alpha)^2+(1-\sin^2\alpha)}=\pi\int_0^{\infty}\frac{dt}{(t+\sin\alpha)^2+\cos^2\alpha}$$

$$=\frac{\pi}{\cos\alpha}\left[\tan^{-1}\frac{t+\sin\alpha}{\cos\alpha}\right]_0^{\infty}=\frac{\pi}{\cos\alpha}\left[\tan^{-1}\frac{\tan\frac{x}{2}+\sin\alpha}{\cos\alpha}\right]_0^{\infty}$$

$$=\frac{\pi}{\cos\alpha}\left[\frac{\pi}{2}-\tan^{-1}(\tan\alpha)\right]=\frac{\pi}{\cos\alpha}\left(\frac{\pi}{2}-\alpha\right)=\frac{\pi(\pi-2\alpha)}{2\cos\alpha}$$

Long Answer Questions-II (OIQ)

[6 Mark]

Q.1. Find: $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$

Ans.

Let $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$

Putting

$\sqrt{x} = \cos\theta,\ i.e.,\ x = \cos^2\theta \quad \Rightarrow \quad \theta = \cos^{-1}\sqrt{x}$ and $dx = -2\cos\theta\sin\theta\, d\theta$, we get

$$I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,(-2\sin\theta\cos\theta)d\theta$$

$$= -2\int \sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\,(\sin\theta\cos\theta)\,d\theta = -2\int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta\right)d\theta$$

$$= -2\int 2\sin^2\frac{\theta}{2}\cos\theta\, d\theta = -2\int (1-\cos\theta)\cos\theta\, d\theta$$

$$= -2\int (1-\cos\theta)\cos\theta\,.\, d\theta = -2\int (\cos\theta - \cos^2\theta).\, d\theta$$

$$= -2\int \cos\theta + \int 2\cos^2\theta.\, d\theta = -2\sin\theta + \int (1+\cos 2\theta).\, d\theta$$

$$= -2\sin\theta + \int 1.d\theta + \int \cos 2\theta.\, d\theta = -2\sin\theta + \theta + \frac{\sin 2\theta}{2} + C$$

$$= -2\sqrt{1-\cos^2\theta} + \theta + \frac{2\sqrt{1-\cos^2\theta}.\,\cos\theta}{2} + C = -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$

Q.2. Find: $\int \frac{x^2}{(x\sin x + \cos x)^2}\, dx$

Ans.

Let $I = \int \frac{x^2 dx}{(x \sin x + \cos x)^2} dx = \int \frac{x \cos x}{(x \sin x + \cos x)^2} \cdot \frac{x}{\cos x} dx$

Integrating by parts, taking $\frac{x}{\cos x}$ as the first function and $\frac{x \cos x}{(x \sin x + \cos x)^2}$ as

the second function, we get

$$I = \frac{x}{\cos x} \int \frac{x \cos x}{(x \sin x + \cos x)^2} \cdot dx - \int \left[\frac{d}{dx} \left(\frac{x}{\cos x} \right) \int \left(\frac{x \cos x}{x \sin x + \cos x} \right) \cdot dx \right] dx$$

Now, let us first evaluate $\int \frac{x \cos x dx}{(x \sin x + \cos x)^2}$

Putting $(x \sin x + \cos x) = t$, then $(\sin x + x \cos x - \sin x) dx = dt$ *i.e.*, $x \cos x \, dx = dt$, we get

$$\int \frac{x \cos x}{(x \sin x + \cos x)^2} dx = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{(x \sin x + \cos x)}$$

Hence, $I = \frac{x}{\cos x} \cdot \frac{-1}{(x \sin x + \cos x)} - \int \frac{\cos x + x \sin x}{\cos^2 x} \times \frac{-1}{(x \sin x + \cos x)} \cdot dx$

$$= \frac{x}{\cos x} \times \frac{-1}{(x \sin x + \cos x)} + \int \sec^2 x . \, dx$$

$$= \frac{-x}{\cos x (x \sin x + \cos x)} + \tan x + C$$

$$= \frac{-x + x \sin^2 x + \sin x \cos x}{\cos x (x \sin x + \cos x)} + C$$

$$= \frac{\cos x (\sin x - x \cos x)}{\cos x (x \sin x + \cos x)} + C$$

$$\int \frac{x^2 dx}{(x \sin x + \cos x)^2} = \frac{(\sin x - x \cos x)}{(x \sin x + \cos x)} + C$$