

Long Answer Question-II (PYQ)

[6 marks]

Q.1. Using the method of integration, find the area of the region bounded by the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$.

Ans.

Given lines are

$$3x - 2y + 1 = 0 \quad \dots(i)$$

$$2x + 3y - 21 = 0 \quad \dots(ii)$$

$$x - 5y + 9 = 0 \quad \dots(iii)$$

For intersection of (i) and (ii)

Applying $(i) \times 3 + (ii) \times 2$, we get

$$9x - 6y + 3 + 4x + 6y - 42 = 0$$

$$\Rightarrow 13x - 39 = 0 \Rightarrow x = 3$$

Putting it in (i), we get

$$9 - 2y + 1 = 0$$

$$\Rightarrow 2y = 10 \Rightarrow y = 5$$

Intersection point of (i) and (ii) is (3, 5)

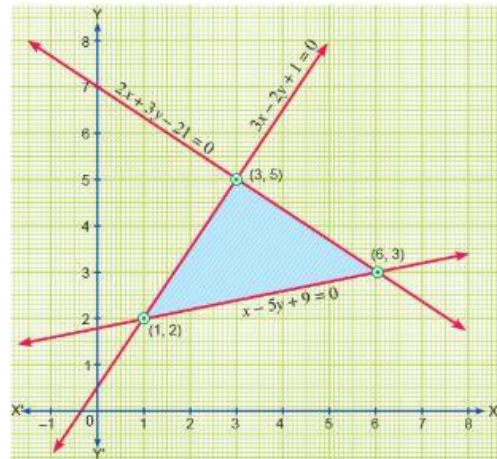
For intersection of (ii) and (iii)

Applying $(ii) - (iii) \times 2$, we get

$$2x + 3y - 21 - 2x + 10y - 18 = 0$$

$$\Rightarrow 13y - 39 = 0 \Rightarrow y = 3$$

Putting $y = 3$ in (ii), we get



$$2x + 9 - 21 = 0$$

$$\Rightarrow 2x - 12 = 0 \Rightarrow x = 6$$

Intersection point of (ii) and (iii) is (6, 3)

For intersection of (i) and (iii)

Applying (i) - (iii) $\times 3$, we get

$$3x - 2y + 1 - 3x + 15y - 27 = 0$$

$$\Rightarrow 13y - 26 = 0 \Rightarrow y = 2$$

Putting $y = 2$ in (i), we get

$$3x - 4 + 1 = 0 \Rightarrow x = 1$$

Intersection point of (i) and (iii) is (1, 2).

With the help of point of intersection we draw the graph of lines (i), (ii) and (iii)

Shaded region is required region.

$$\begin{aligned} \therefore \text{Area of required region} &= \int_1^3 \frac{3x+1}{2} dx + \int_3^6 \frac{-2x+21}{3} dx - \int_1^6 \frac{x+9}{5} dx \\ &= \frac{3}{2} \int_1^3 x dx + \frac{1}{2} \int_1^3 dx - \frac{2}{3} \int_3^6 x dx + 7 \int_3^6 dx - \frac{1}{5} \int_1^6 x dx - \frac{9}{5} \int_1^6 dx \\ &= \frac{3}{2} \left[\frac{x^2}{2} \right]_1^3 + \frac{1}{2} [x]_1^3 - \frac{2}{3} \left[\frac{x^2}{2} \right]_3^6 + 7[x]_3^6 - \frac{1}{5} \left[\frac{x^2}{2} \right]_1^6 - \frac{9}{5} [x]_1^6 \\ &= \frac{3}{4} (9-1) + \frac{1}{2} (3-1) - \frac{2}{6} (36-9) + 7(6-3) - \frac{1}{10} (36-1) - \frac{9}{5} (6-1) \\ &= 6 + 1 - 9 + 21 - \frac{7}{2} - 9 = 10 - \frac{7}{2} = \frac{20-7}{2} = \frac{13}{2} \text{ sq sq units.} \end{aligned}$$

Q.2. Find the area of the region $\{(x, y) : x^2 + y^2 \leq 4, x + y \geq 2\}$.

Ans.

$$\text{Let } R = \{(x, y) : x^2 + y^2 \leq 4, x + y \geq 2\}$$

$$\Rightarrow R = \{(x, y) : x^2 + y^2 \leq 4\} \cap \{(x, y) : x + y \geq 2\}$$

i.e., $R = R^1 \cap R^2$, where

$$R_1 = \{(x, y) : x^2 + y^2 \leq 4\} \text{ and } R_2 = \{(x, y) : x + y \geq 2\}$$

For region R_1

Obviously $x^2 + y^2 = 4$ is a circle having centre at $(0,0)$ and radius 2.

Since $(0,0)$ satisfy $x^2 + y^2 \leq 4$. Therefore region R_1 is the region lying interior of circle $x^2 + y^2 = 4$

For region R

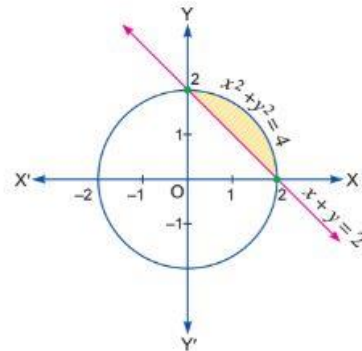
| | | |
|-----|---|---|
| x | 0 | 2 |
| y | 2 | 0 |

$x + y = 2$ is a straight line passing through $(0, 2)$ and $(2, 0)$. Since $(0, 0)$ does not satisfy $x + y \geq 2$ therefore R_2 is that region which does not contain origin $(0, 0)$ *i.e.*, above the line $x + y = 2$

Hence, shaded region is required area.

Now, area of required region

$$\begin{aligned} &= \int_0^2 \sqrt{4-x^2} dx - \int_0^2 (2-x) dx \\ &= \left[\frac{1}{2} x \sqrt{4-x^2} + \frac{1}{2} 4 \sin^{-1} \left(\frac{x}{2} \right) \right]_0^2 - 2[x]_0^2 + \left[\frac{x^2}{2} \right]_0^2 \\ &= [2 \sin^{-1} 1 - 0] - 2[2 - 0] + \left[\frac{4}{2} - 0 \right] \\ &= 2 \times \frac{\pi}{2} - 4 + 2 = (\pi - 2) \text{ sq units} \end{aligned}$$



Q.3. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

Ans.

Given curves are

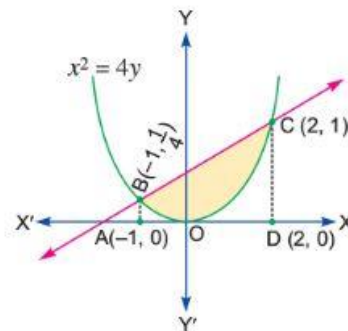
$$x^2 = 4y \quad \dots(i)$$

and $x = 4y - 2 \quad \dots(ii)$

Obviously, curve (i) is a parabola with vertex at (0, 0) and axis along +ve direction of y-axis, while equation (ii) represents a straight line.

For points of intersection, we solve (i) and (ii) as

$$\begin{aligned} x^2 &= x + 2 \\ \Rightarrow x^2 - x - 2 &= 0 & \Rightarrow x^2 - 2x + x - 2 &= 0 \\ \Rightarrow x(x - 2) + 1(x - 2) &= 0 & \Rightarrow (x - 2)(x + 1) &= 0 \\ \Rightarrow x = -1, 2 & \Rightarrow y = \frac{1}{4}, 1 \end{aligned}$$



Hence, coordinates of intersection points of line and parabola are $(-1, \frac{1}{4})$ and $(2, 1)$.

Shaded region is required region.

∴ Required area = area of trap. ABCD - area of ABOCDOA

$$\begin{aligned} &= \int_{-1}^2 \left(\frac{x+2}{4} \right) dx - \int_{-1}^2 \frac{x^2}{4} dx = \frac{1}{4} \left[\frac{(x+2)^2}{2} \right]_{-1}^2 - \frac{1}{4} \left[\frac{x^3}{3} \right]_{-1}^2 \\ &= \frac{1}{8} [16 - 1] - \frac{1}{12} (8 + 1) \\ &= \frac{15}{8} - \frac{9}{12} = \frac{27}{24} = \frac{9}{8} \text{ sq units.} \end{aligned}$$

Q.3. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

Ans.

Given curves are

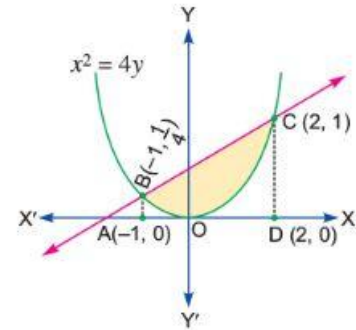
$$x^2 = 4y \quad \dots(i)$$

and $x = 4y - 2 \quad \dots(ii)$

Obviously, curve (i) is a parabola with vertex at (0, 0) and axis along +ve direction of y-axis, while equation (ii) represents a straight line.

For points of intersection, we solve (i) and (ii) as

$$\begin{aligned} x^2 &= x + 2 \\ \Rightarrow x^2 - x - 2 &= 0 & \Rightarrow x^2 - 2x + x - 2 &= 0 \\ \Rightarrow x(x - 2) + 1(x - 2) &= 0 & \Rightarrow (x - 2)(x + 1) &= 0 \\ \Rightarrow x = -1, 2 & \Rightarrow y = \frac{1}{4}, 1 \end{aligned}$$



Hence, coordinates of intersection points of line and parabola are $(-1, \frac{1}{4})$ and $(2, 1)$.

Shaded region is required region.

\therefore Required area = area of trap. $ABCD$ - area of $ABOCDOA$

$$\begin{aligned} &= \int_{-1}^2 \left(\frac{x+2}{4} \right) dx - \int_{-1}^2 \frac{x^2}{4} dx = \frac{1}{4} \left[\frac{(x+2)^2}{2} \right]_{-1}^2 - \frac{1}{4} \left[\frac{x^3}{3} \right]_{-1}^2 \\ &= \frac{1}{8} [16 - 1] - \frac{1}{12} (8 + 1) \\ &= \frac{15}{8} - \frac{9}{12} = \frac{27}{24} = \frac{9}{8} \text{ sq units.} \end{aligned}$$

Q.4. Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$.

Ans.

Given parabolas are $y^2 = 4ax \dots(i)$, $x^2 = 4ay \dots(ii)$

Obviously, curve (i) is right handed parabola having vertex at (0, 0), while curve (ii) is upward parabola having vertex at (0, 0).

Shaded region is required region.

For coordinate of intersection point A, (i) and (ii) are solved as

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow x^4 = 64a^3x$$

$$\Rightarrow x(x^3 - 64a^3) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x^3 - 64a^3 = 0$$

$$\Rightarrow x = 4a \quad \text{and} \quad y = 4a$$

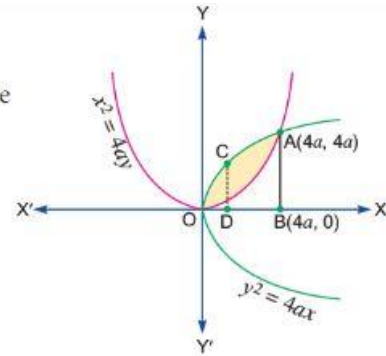
Hence, coordinate of A $\equiv (4a, 4a)$

Therefore, area of required region = area of OCABO - area of ODABO

$$= \int_0^{4a} \sqrt{4ax} \, dx - \int_0^{4a} \frac{x^2}{4a} \, dx \quad \Rightarrow \quad A = \left[\frac{4\sqrt{a}}{3} x^{3/2} - \frac{x^3}{12a} \right]_0^{4a}$$

$$\Rightarrow A = \frac{4\sqrt{a}}{3} (4a)^{3/2} - \frac{(4a)^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} \text{ sq units.}$$



Q.5. Find the area of the region enclosed between the two circles: $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

Ans.

Equations of the given circles are

$$x^2 + y^2 = 4 \quad \dots(i)$$

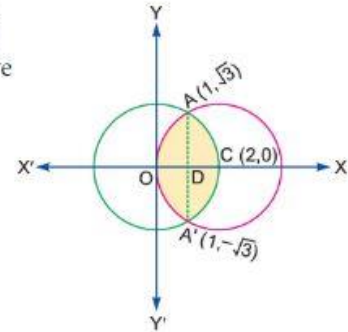
and $(x - 2)^2 + y^2 = 4 \quad \dots(ii)$

Equation (i) is a circle with centre O at the origin and radius 2. Equation (ii) is a circle with centre $C(2, 0)$ and radius 2. Solving equations (i) and (ii), we have

$$(x - 2)^2 + y^2 = x^2 + y^2$$

or $x^2 - 4x + 4 + y^2 = x^2 + y^2$

or $x = 1$ which gives $y = \pm\sqrt{3}$



Thus, the points of intersection of the given circles are $A(1, \sqrt{3})$ and $A'(1, -\sqrt{3})$ as shown in the fig.

Required area of the enclosed region $OACA'O$ between circles

$$= 2 \text{ [area of the region } ODCAO]$$

$$= 2 \text{ [area of the region } ODAO + \text{area of the region } DCAD]$$

$$= 2 \left[\int_0^1 y \, dx + \int_1^2 y \, dx \right]$$

$$= 2 \left[\int_0^1 \sqrt{4 - (x - 2)^2} \, dx + \int_1^2 \sqrt{4 - x^2} \, dx \right]$$

$$= 2 \left[\frac{1}{2}(x - 2)\sqrt{4 - (x - 2)^2} + \frac{1}{2} \times 4\sin^{-1} \left(\frac{x-2}{2} \right) \right]_0^1 + 2 \left[\frac{1}{2}x\sqrt{4 - x^2} + \frac{1}{2} \times 4\sin^{-1} \frac{x}{2} \right]_1^2$$

$$= \left[(x - 2)\sqrt{4 - (x - 2)^2} + 4\sin^{-1} \left(\frac{x-2}{2} \right) \right]_0^1 + \left[x\sqrt{4 - x^2} + 4\sin^{-1} \frac{x}{2} \right]_1^2$$

$$= \left[(-\sqrt{3} + 4\sin^{-1} \left(\frac{-1}{2} \right)) - 4\sin^{-1}(-1) \right] + \left[4\sin^{-1} 1 - \sqrt{3} - 4\sin^{-1} \frac{1}{2} \right]$$

$$= \left[\left(-\sqrt{3} - 4 \times \frac{\pi}{6} \right) + 4 \times \frac{\pi}{2} \right] + \left[4 \times \frac{\pi}{2} - \sqrt{3} - 4 \times \frac{\pi}{6} \right]$$

$$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi \right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3} \right) = \frac{8\pi}{3} - 2\sqrt{3} \text{ sq units .}$$

Q.6. Using integration find the area of the region

$$\{(x, y) : 25x^2 + 9y^2 \leq 225 \text{ and } 5x + 3y \geq 15\}.$$

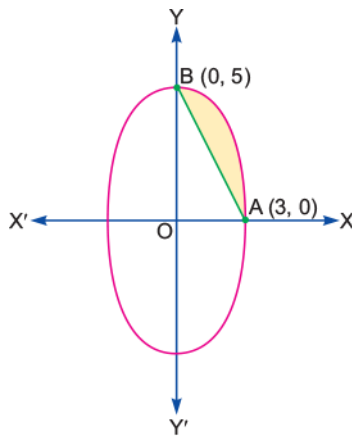
Ans.

The given region is $\{(x, y) : 25x^2 + 9y^2 \geq 225 \text{ and } 5x + 3y \geq 15\}$

Its corresponding equations are

$$25x^2 + 9y^2 = 225 \Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1 \quad \dots(i)$$

and $5x + 3y = 15 \quad \dots(ii)$



Obviously; curve (i) is an ellipse with centre at (0, 0) and major axis along y-axis while (ii) is a straight line. Intersection points of (i) and (ii) are (0, 5) and (3, 0).

Shaded region is required region and is given by

$$\begin{aligned} & \int_0^3 5\sqrt{1 - \frac{x^2}{9}} dx - \int_0^3 \frac{15 - 5x}{3} dx \\ &= \frac{5}{3} \int_0^3 \sqrt{9 - x^2} dx - 5 \int_0^3 dx + \frac{5}{3} \int_0^3 x dx \\ &= \left[\frac{5x}{6} \sqrt{9 - x^2} + \frac{15}{2} \sin^{-1} \frac{x}{3} \right]_0^3 - 5/x \Big|_0^3 + \left[\frac{5x^2}{6} \right]_0^3 \\ &= \frac{15}{2} \cdot \frac{\pi}{2} - 15 + \frac{15}{2} = \frac{15}{2} \left(\frac{\pi}{2} - 1 \right) \text{ sq units.} \end{aligned}$$

Q.7. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

Ans.

The given equations are

$$y = x \quad \dots(i)$$

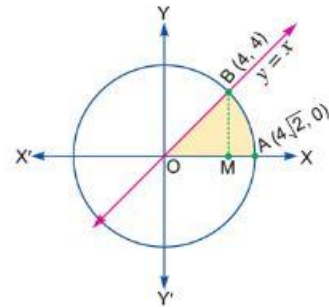
and $x^2 + y^2 = 32 \quad \dots(ii)$

Solving (i) and (ii), we find that the line and the circle meet at $B(4, 4)$ in the first quadrant. Draw perpendicular BM to the x -axis.

Therefore, the required area = area of the region $OBMO$ + area of the region $BMAB$.

Now, the area of the region $OBMO$

$$= \int_0^4 y \, dx = \int_0^4 x \, dx = \frac{1}{2} [x^2]_0^4 = 8 \quad \dots(iii)$$



Again, the area of the region $BMAB$

$$\begin{aligned} &= \int_4^{4\sqrt{2}} y \, dx = \int_4^{4\sqrt{2}} \sqrt{32 - x^2} \, dx = \left[\frac{1}{2} x \sqrt{32 - x^2} + \frac{1}{2} \times 32 \times \sin^{-1} \frac{x}{4\sqrt{2}} \right]_4^{4\sqrt{2}} \\ &= \left(\frac{1}{2} 4\sqrt{2} \times 0 + \frac{1}{2} \times 32 \times \sin^{-1} 1 \right) - \left(\frac{1}{2} 4 \sqrt{32 - 16} + \frac{1}{2} \times 32 \times \sin^{-1} \frac{1}{\sqrt{2}} \right) \\ &= 8\pi - (8 + 4\pi) = 4\pi - 8 \quad \dots(iv) \end{aligned}$$

Adding (iii) and (iv), we get the required area = 4π sq units.

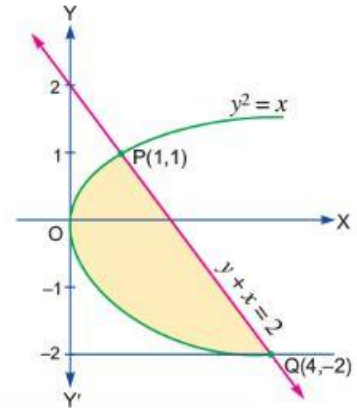
Q.8. Find the area enclosed by the parabola $y^2 = x$ and line $y + x = 2$.

Ans.

To get the point of intersection, we have to solve the equation of line $y + x = 2$ and parabola $y^2 = x$.

On solving them we find the coordinates of points of intersection as $(4, -2)$ and $(1, 1)$. Drawing perpendiculars from these points on y -axis, we obtain the coordinate as $(0, 1)$ and $(0, -2)$.

$$\begin{aligned} \text{Thus, required area} &= \int_{-2}^1 (2 - y - y^2) dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^1 \\ &= \left(2 - \frac{1}{2} - \frac{1}{3} \right) - \left(-4 - \frac{4}{2} + \frac{8}{3} \right) \\ &= 2 - \frac{5}{6} + 6 - \frac{8}{3} = 8 - \frac{21}{6} = \frac{27}{6} \\ &= \frac{9}{2} \text{ sq units.} \end{aligned}$$



Q.9. Using integration, find the area of the region bounded by the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$.

Ans.

Given curves are

$$y^2 = 4x \dots (i)$$

$$4x^2 + 4y^2 = 9 \dots (ii)$$

$$\Rightarrow x^2 + y^2 = \left(\frac{3}{2}\right)^2$$

Obviously, curve (i) is right handed parabola having vertex at (0, 0) and axis along +ve direction of x-axis while curve (ii) is a circle having centre at (0, 0) and radius $\frac{3}{2}$.

Shaded region is required bounded region, which is symmetrical about x-axis.

For coordinate of intersection points 'A' or 'B'.

$$\Rightarrow 4x^2 + 16x - 9 = 0$$

$$\Rightarrow 4x^2 + 18x - 2x - 9 = 0$$

$$\Rightarrow 2x(2x + 9) - 1(2x + 9) = 0$$

$$\Rightarrow (2x - 1)(2x + 9) = 0$$

$$\Rightarrow x = \frac{1}{2} \text{ or } x = -\frac{9}{2} \text{ (not possible)}$$

$$\text{i.e., } x = \frac{1}{2} \quad \therefore y^2 = 4 \times \frac{1}{2} = 2$$

Hence, area of required region = 2 [Area $OADO$ + Area $DACD$]

$$\begin{aligned}
 &= 2 \left[\int_0^{1/2} 2\sqrt{x} dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2} dx \right] \\
 &= 4 \cdot \frac{2}{3} [x^{3/2}]_0^{1/2} + \left[\frac{x}{2} \sqrt{\frac{9}{4} - x^2} + \frac{1}{2} \cdot \frac{9}{4} \sin^{-1} \frac{2x}{3} \right]_{1/2}^{3/2} \\
 &= \frac{8}{3} \left(\frac{1}{2\sqrt{2}} - 0 \right) + \left[\left\{ \frac{9}{4} \sin^{-1}(1) \right\} - \left\{ \frac{1}{\sqrt{2}} + \frac{9}{4} \sin^{-1} \left(\frac{1}{3} \right) \right\} \right] \\
 &= \frac{2\sqrt{2}}{3} + \left[\frac{9\pi}{8} - \frac{1}{\sqrt{2}} - \frac{9}{4} \sin^{-1} \left(\frac{1}{3} \right) \right] \text{ sq units.} \\
 &= \frac{2\sqrt{2}}{3} - \frac{1}{\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \left(\frac{1}{3} \right)
 \end{aligned}$$

[**Note:** Equation of circle in standard form is

$$(x - \alpha)^2 + (y - \beta)^2 = r^2, \text{ where } (\alpha, \beta) \text{ is centre and } r \text{ is radius.}]$$

Q.10. Find the area of that part of the circle $x^2 + y^2 = 16$, which is exterior to the parabola $y^2 = 6x$.

Ans.

Given curves are

$$x^2 + y^2 = 16 \quad \dots(i)$$

$$y^2 = 6x \quad \dots(ii)$$

Obviously curve (i) is a circle having centre at (0, 0) and radius 4 unit.

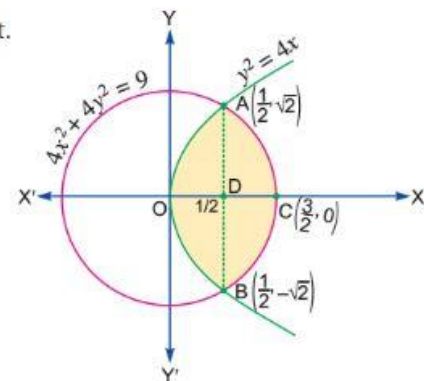
While curve (ii) is right handed parabola having vertex at (0, 0) and axis along +ve direction of x-axis.

Required part in the shaded region.

Now, for intersection point of curve (i) & (ii)

$$x^2 + 6x = 16$$

$$\Rightarrow x^2 + 6x - 16 = 0$$



$$\Rightarrow x^2 + 8x - 2x - 16 = 0$$

$$\Rightarrow x(x + 8) - 2(x + 8) = 0$$

$$\Rightarrow (x + 8)(x - 2) = 0$$

$$\Rightarrow x = -8 \text{ or } 2 \quad [\because x = -8 \text{ is not possible as } y^2 \text{ is +ve}]$$

$$\therefore x = 2$$

$$\therefore y = \pm \sqrt{12} = \pm 2\sqrt{3}$$

Hence, coordinate of B is $(2, 2\sqrt{3})$.

Since, shaded region *i.e.*, required part is symmetrical about x -axis.

Therefore, area of required part

$$= 2 [\text{Area } OBEDO]$$

$$= 2 [\text{Area } DEBFD - \text{Area } OBFO]$$

$$= 2 \left[\int_{-4}^2 \sqrt{16 - x^2} \, dx - \int_0^2 \sqrt{6x} \, dx \right]$$

$$= 2 \left[\frac{x}{2} \sqrt{16 - x^2} + \frac{16}{2} \sin^{-1} \frac{x}{4} \right]_{-4}^2 - 2\sqrt{6} \cdot \frac{2}{3} [x^{3/2}]_0^2$$

$$= \left[\left(2 \times \sqrt{12} + 16 \sin^{-1} \frac{1}{2} \right) - (-4) \times 0 + 16 \sin^{-1}(-1) \right] - \frac{4\sqrt{6}}{3} [2\sqrt{2} - 0]$$

$$= \left(\frac{8}{3} + 8 \right) \pi + \left(4 - \frac{16}{3} \right) \sqrt{3}$$

$$= \left(\frac{32}{3} \pi - \frac{4}{3} \sqrt{3} \right) \text{ sq units.}$$

Q.11. Using integration, find the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

Ans.

Given, $x + 2y = 2$... (i)

$y - x = 1$... (ii)

$2x + y = 7$... (iii)

On solving equation (i) and (ii), we get

Coordinate of $A \equiv (0, 1)$.

Similarly,

By solving (ii) and (iii), we get

Co-ordinate of $B \equiv (2, -3)$

and by solving (i) and (iii), we get

Co-ordinate of $C \equiv (4, -1)$

Shaded region is required region.

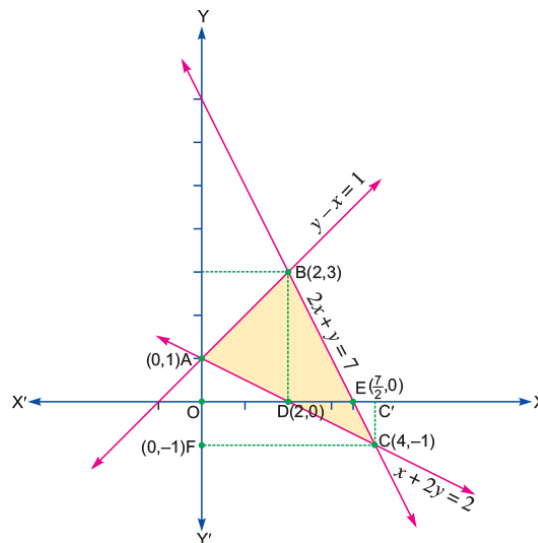
\therefore Area of required region

$$= \int_{-1}^3 \frac{7-y}{2} dy - \int_{-1}^1 (2-2y) dy - \int_1^3 (y-1) dy$$

$$= \frac{1}{2} \left[7y - \frac{y^2}{2} \right]_{-1}^3 - [2y - y^2]_{-1}^1 - \left[\frac{y^2}{2} - y \right]_1^3$$

$$= \frac{1}{2} \left(21 - \frac{9}{2} + 7 + \frac{1}{2} \right) - (2 - 1 + 2 + 1) - \left(\frac{9}{2} - 3 - \frac{1}{2} + 1 \right)$$

$$= 12 - 4 - 2 = 6 \text{ sq units}$$



Q.12. Find the area of the region $\{(x, y) : y^2 \leq 6ax \text{ and } x^2 + y^2 \leq 16a^2\}$, using method of integration.

Ans.

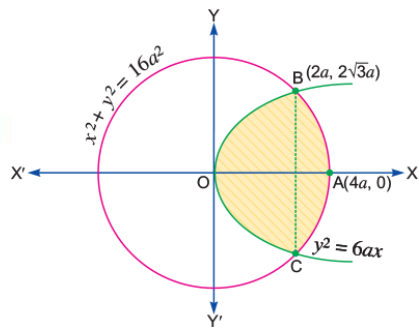
Corresponding curves of given region

$\{(x, y): y^2 \leq 6ax \text{ and } x^2 + y^2 \leq 16a^2\}$ are

$$x^2 + y^2 = 16a^2 \quad \dots(i)$$

$$y^2 = 6ax \quad \dots(ii)$$

Obviously, curve (i) is a circle having centre (0, 0) and radius 4a. While curve (ii) is right handed parabola having vertex at (0, 0) and axis along +ve direction of x-axis.



Obviously, shaded region $OCAB$ is area represented by

$$y^2 \leq 6ax$$

and $x^2 + y^2 \leq 16a^2$

Now, intersection point of curve (i) and (ii)

$$x^2 + 6ax = 16a^2 \quad [\text{Putting the value of } y^2 \text{ in (i)}]$$

$$\Rightarrow x^2 + 6ax - 16a^2 = 0 \quad \Rightarrow x^2 + 8ax - 2ax - 16a^2 = 0$$

$$\Rightarrow x(x + 8a) - 2a(x + 8a) = 0 \quad \Rightarrow (x + 8a)(x - 2a) = 0$$

$$\Rightarrow x = 2a, -8a$$

$$\Rightarrow x = 2a \quad [\because x = -8a \text{ is not possible as } y^2 \text{ is +ve}]$$

$$\therefore y = 2\sqrt{3}a$$

Since, shaded region is symmetrical about x -axis

\therefore Required area = 2 [Area of $OABO$]

$$\begin{aligned} &= 2 \left[\int_0^{2a} \sqrt{6ax} \, dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2} \, dx \right] \\ &= 2 \left[\sqrt{6a} \int_0^{2a} \sqrt{x} \, dx + \int_{2a}^{4a} \sqrt{(4a)^2 - x^2} \, dx \right] \\ &= 2\sqrt{6a} \times \frac{2}{3} [x^{3/2}]_0^{2a} + 2 \left[\frac{x}{2} \sqrt{16a^2 - x^2} + \frac{1}{2} 16a^2 \sin^{-1} \frac{x}{4a} \right]_{2a}^{4a} \\ &= \frac{4\sqrt{6a}}{3} [(2a)^{3/2} - 0] + 2 \left[(0 + 4a^2\pi) - \left(2\sqrt{3}a^2 + \frac{4a^2\pi}{3} \right) \right] \\ &= \frac{8a^2\sqrt{12}}{3} + 8a^2\pi - 4\sqrt{3}a^2 - \frac{8a^2\pi}{3} \\ &= \frac{16}{3}\sqrt{3}a^2 + \frac{16a^2\pi}{3} - 4\sqrt{3}a^2 \\ &= \frac{4\sqrt{3}}{3}a^2 + \frac{16a^2\pi}{3} \end{aligned}$$

Q.13. Find the area of the region enclosed between the two circles $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$.

Ans.

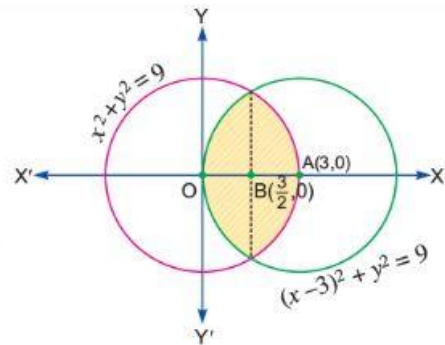
The two circles are re-arranged and expressed as

$$y^2 = 9 - x^2 \quad \dots (i)$$

$$y^2 = 9 - (x - 3)^2 \quad \dots (ii)$$

Obviously, circle (i) have centre $(0, 0)$ and radius 3 while (ii) have centre $(3, 0)$ and radius 3.

To find the point of intersection of the circles we equate y^2 .



To find the point of intersection of the circles we equate y^2 .

$$\Rightarrow 9 - x^2 = 9 - (x - 3)^2$$

$$\Rightarrow 9 - x^2 = 9 - x^2 - 9 + 6x$$

$$\Rightarrow x = \frac{3}{2}$$

The circles are shown in the figure and the shaded area is the required area.

Now, area of shaded region

$$\begin{aligned} &= 2 \left[\int_0^{\frac{3}{2}} \sqrt{9 - (x - 3)^2} dx + \int_{\frac{3}{2}}^3 \sqrt{9 - x^2} dx \right] \\ &= 2 \left[\frac{x-3}{2} \sqrt{9 - (x-3)^2} + \frac{9}{2} \sin^{-1} \frac{x-3}{3} \right]_0^{\frac{3}{2}} + 2 \left[\frac{x}{2} \sqrt{9 - x^2} + \frac{9}{2} \sin^{-1} \frac{x}{3} \right]_{\frac{3}{2}}^3 \\ &= 2 \left[\frac{-3}{4} \sqrt{9 - \frac{9}{4}} + \frac{9}{2} \sin^{-1} \left(-\frac{1}{2}\right) - \frac{9}{2} \sin^{-1}(-1) \right] + 2 \left[\frac{9}{2} \sin^{-1} 1 - \frac{3}{4} \sqrt{9 - \frac{9}{4}} - \frac{9}{2} \sin^{-1} \frac{1}{2} \right] \\ &= 2 \left[\frac{-3}{4} \cdot \frac{3\sqrt{3}}{2} - \frac{9}{2} \cdot \frac{\pi}{6} + \frac{9}{2} \cdot \frac{\pi}{2} \right] + 2 \left[\frac{9}{2} \cdot \frac{\pi}{2} - \frac{3}{4} \cdot \frac{3\sqrt{3}}{2} - \frac{9}{2} \cdot \frac{\pi}{6} \right] \\ &= 2 \left[-\frac{9\sqrt{3}}{8} - \frac{3\pi}{4} + \frac{9\pi}{4} + \frac{9\pi}{4} - \frac{9}{8}\sqrt{3} - \frac{3\pi}{4} \right] \\ &= 2 \left[\frac{-9\sqrt{3}}{4} - \frac{6\pi}{4} + \frac{18\pi}{4} \right] \\ &= 2 \left[-\frac{9\sqrt{3}}{4} + \frac{12\pi}{4} \right] = 6\pi - \frac{9\sqrt{3}}{2} \text{ sq units.} \end{aligned}$$

Q.14. Find the area of the region included between the parabola $4y = 3x^2$ and the line $3x - 2y + 12 = 0$.

Ans.

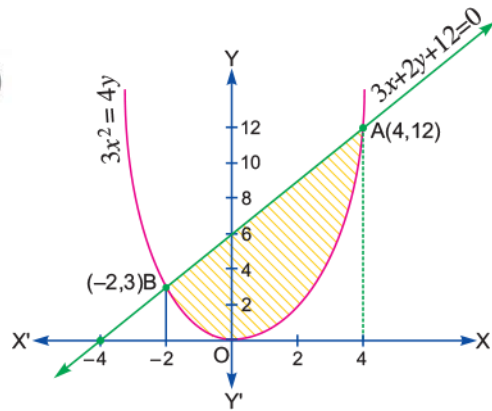
Given equation of parabola $4y = 3x^2 \Rightarrow y = \frac{3x^2}{4}$... (i)

and the line $3x - 2y + 12 = 0$

$$\Rightarrow \frac{3x+12}{2} = y \quad \dots (ii)$$

The line intersect the parabola at $(-2, 3)$ and $(4, 12)$.

Hence, the required area will be the shaded region.



$$\text{Required area} = \int_{-2}^4 \frac{3x+12}{2} dx - \int_{-2}^4 \frac{3x^2}{4} dx$$

$$= \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4} \right]_{-2}^4$$

$$= (12 + 24 - 16) - (3 - 12 + 2)$$

$$= 20 + 7$$

$$= 27 \text{ square units.}$$

Q.15. Using integration, find the area of the region

$$\{(x, y) : 9x^2 + y^2 \leq 36 \text{ and } 3x + y \geq 6.\}$$

Ans.

Given region is $\{(x, y) : 9x^2 + y^2 \leq 36 \text{ and } 3x + y \geq 6\}$

We draw the curves corresponding to equations

$$9x^2 + y^2 = 36 \quad \text{or}$$

$$\frac{x^2}{4} + \frac{y^2}{36} = 1 \text{ and } 3x + y = 6$$

The curves intersect at $(2, 0)$ and $(0, 6)$

\therefore Shaded area is enclosed by the two curves and is

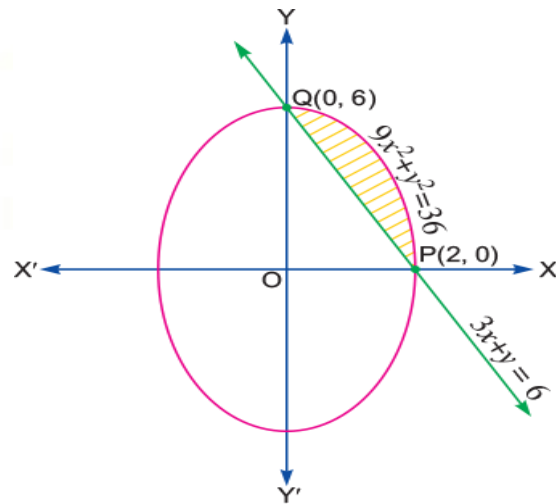
$$= \int_0^2 \sqrt{36 \left(1 - \frac{x^2}{4}\right)} dx - \int_0^2 (6 - 3x) dx$$

$$= 6 \left[\frac{x}{4} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} - 2x + \frac{x^2}{2} \right]_0^2$$

$$= 6 \left[\frac{2}{4} \sqrt{4 - 4} + \frac{4}{2} \sin^{-1} \frac{2}{2} - 4 + \frac{4}{2} - 0 \right]$$

$$= 6 \left[2 \frac{\pi}{2} - 2 \right]$$

$$= 6(\pi - 2) \text{ square units}$$

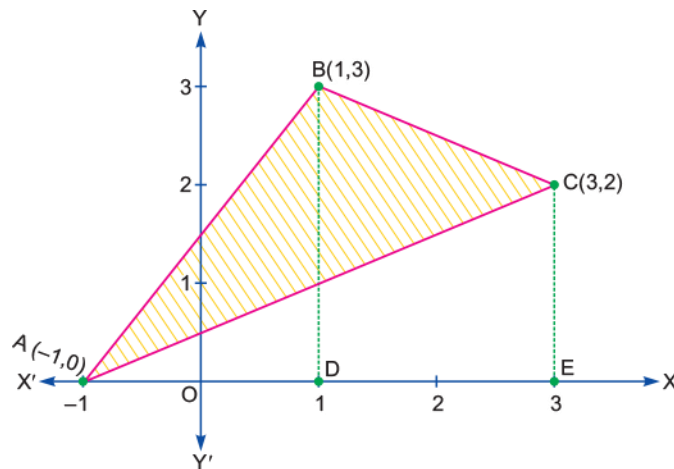


Q.16. Using integration, find the area of the triangle ABC with vertices as $A(-1, 0)$, $B(1, 3)$ and $C(3, 2)$.

Ans.

We mark the points on the graph and get the triangle ABC as shown in the figure

Required area of triangle = area of $DABD$ + area of trapezium $EDEC$ – area of $DACE$



$$\text{Equation of line } AB \Rightarrow y = \frac{3}{2}(x + 1)$$

$$\text{Equation of line } BC \Rightarrow y = -\frac{x}{2} + \frac{7}{2}$$

$$\text{Equation of line } AC \Rightarrow y = \frac{x}{2} + \frac{1}{2}$$

∴ Area of

$$\begin{aligned} \Delta ABC &= \int_{-1}^1 \left(\frac{3}{2}x + \frac{3}{2} \right) dx + \int_1^3 \left(-\frac{x}{2} + \frac{7}{2} \right) dx - \int_{-1}^3 \left(\frac{x}{2} + \frac{1}{2} \right) dx \\ &= \left[\frac{3x^2}{4} + \frac{3}{2}x \right]_{-1}^1 + \left[-\frac{x^2}{4} + \frac{7}{2}x \right]_1^3 - \left[\frac{x^2}{4} + \frac{x}{2} \right]_{-1}^3 \\ &= \left(\frac{3}{4} + \frac{3}{2} - \frac{3}{4} + \frac{3}{2} \right) + \left(-\frac{9}{4} + \frac{21}{2} + \frac{1}{4} - \frac{7}{2} \right) - \left(\frac{9}{4} + \frac{3}{2} - \frac{1}{4} + \frac{1}{2} \right) \\ &= 3 + \frac{-9+42+1-14}{4} - \left(\frac{9+6-1+2}{4} \right) = 3 + 5 - 4 = 4 \text{ sq units} \end{aligned}$$

Q.17. Using integration, find the area of the following region:

$$\{(x, y) : |x - 1| \leq y \leq \sqrt{5 - x^2}\}$$

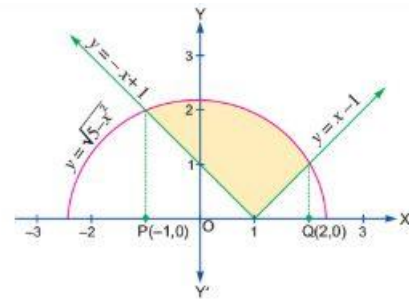
Ans.

We have provided

$$\{(x, y) : |x - 1| \leq y \leq \sqrt{5 - x^2}\}$$

Equation of curve is $y = \sqrt{5 - x^2}$ or $y^2 + x^2 = 5$, which is a circle with centre at $(0, 0)$ and radius $\frac{\sqrt{5}}{2}$.

Equation of line is $y = |x - 1|$



Consider, $y = x - 1$ and $y = \sqrt{5 - x^2}$

Eliminating y , we get

$$x - 1 = \sqrt{5 - x^2}$$

$$\Rightarrow x^2 + 1 - 2x = 5 - x^2$$

$$\Rightarrow 2x^2 - 2x - 4 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x - 2)(x + 1) = 0$$

$$\Rightarrow x = 2, -1$$

The required area is

$$\begin{aligned} &= \int_{-1}^2 \sqrt{5 - x^2} dx - \int_{-1}^1 (-x + 1) dx - \int_1^2 (x - 1) dx \\ &= \left[\frac{x}{2} \sqrt{5 - x^2} + \frac{5}{2} \sin^{-1} \frac{x}{\sqrt{5}} \right]_{-1}^2 - \left[-\frac{x^2}{2} + x \right]_{-1}^1 - \left[\frac{x^2}{2} - x \right]_1^2 \\ &= \left(1 + \frac{5}{2} \sin^{-1} \frac{2}{\sqrt{5}} \right) + 1 - \frac{5}{2} \sin^{-1} \left(-\frac{1}{\sqrt{5}} \right) - \left(\frac{-1}{2} + 1 + \frac{1}{2} + 1 \right) - \left(2 - 2 - \frac{1}{2} + 1 \right) \\ &= \frac{5}{2} \left(\sin^{-1} \frac{2}{\sqrt{5}} + \sin^{-1} \frac{1}{\sqrt{5}} \right) + 2 - 2 - \frac{1}{2} \\ &= \frac{5}{2} \sin^{-1} \left[\frac{2}{\sqrt{5}} \sqrt{1 - \frac{1}{5}} + \frac{1}{\sqrt{5}} \sqrt{1 - \frac{4}{5}} \right] - \frac{1}{2} \\ &= \frac{5}{2} \left[\sin^{-1} \left(\frac{4}{5} + \frac{1}{5} \right) \right] - \frac{1}{2} \\ &= \frac{5}{2} \sin^{-1} (1) - \frac{1}{2} \\ &= \left(\frac{5\pi}{4} - \frac{1}{2} \right) \text{ sq units.} \end{aligned}$$

Q.18. Find the area of the circle $4x^2 + 4y^2 = 9$, which is interior to the parabola $x^2 = 4y$.

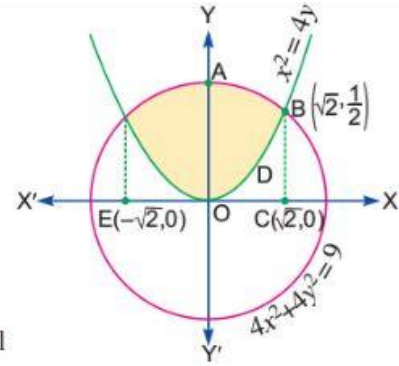
Ans.

Equation of circle and parabola is

$$4x^2 + 4y^2 = 9 \Rightarrow x^2 + y^2 = \frac{9}{4} \quad \dots(i)$$

$$x^2 = 4y \Rightarrow y = \frac{x^2}{4} \quad \dots(ii)$$

Obviously given circle have centre at $(0, 0)$ and radius $\frac{3}{2}$, while given parabola is upward parabola having vertex at $(0, 0)$ and axis along +ve direction of y -axis. Shaded region is required region which is symmetrical along y -axis.



By putting value of equation (ii) in equation (i), we get

$$4x^2 + 4\left(\frac{x^2}{4}\right)^2 = 9$$

$$\Rightarrow x^4 + 16x^2 - 36 = 0$$

$$\Rightarrow (x^2 + 18)(x^2 - 2) = 0$$

$$\Rightarrow x^2 + 18 = 0, \quad x^2 - 2 = 0$$

$$\Rightarrow x = -\sqrt{18}, \quad x = \pm\sqrt{2}$$

$$\Rightarrow x = \pm\sqrt{2} \quad (\because x = -\sqrt{18} \text{ is not possible i.e., intersecting points are } (-\sqrt{2}, 1/2), (\sqrt{2}, 1/2))$$

$$\therefore \text{ Required area} = 2 [\text{area of } OABCO - \text{area of } ODBCO]$$

$$= 2 \left[\int_0^{\sqrt{2}} \sqrt{\frac{9}{4} - x^2} dx - \int_0^{\sqrt{2}} \frac{x^2}{4} dx \right]$$

$$= 2 \left\{ \left[\frac{x}{2} \sqrt{\frac{9}{4} - x^2} + \frac{9}{8} \sin^{-1} \frac{x}{3/2} \right]_0^{\sqrt{2}} - \left[\frac{x^3}{12} \right]_0^{\sqrt{2}} \right\}$$

$$= 2 \left[\frac{\sqrt{2}}{4} + \frac{9}{8} \sin^{-1} \frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{6} \right] = \left(\frac{\sqrt{2}}{6} + \frac{9}{4} \sin^{-1} \frac{2\sqrt{2}}{3} \right) \text{ sq units.}$$

Q.19. Using integration, find the area of the triangle ABC, co-ordinates of whose vertices are A(4, 1), B(6, 6) and C(8, 4)

Ans.

Given triangle ABC , coordinates of whose vertices are $A(4, 1)$, $B(6, 6)$ and $C(8, 4)$.

Equation of AB is given by

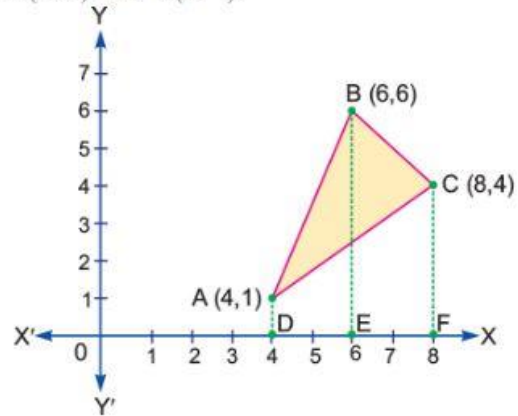
$$y - 6 = \frac{6-1}{6-4}(x - 6) \text{ or } y = \frac{5}{2}x - 9$$

Equation of BC is given by

$$y - 4 = \frac{4-6}{8-6}(x - 8) \text{ or } y = -x + 12$$

Equation of AC is given by

$$y - 4 = \frac{4-1}{8-4}(x - 4) \text{ or } y = \frac{3}{4}x - 2$$



\therefore Area of $\triangle ABC$ = area of trap. $DABE$ + area of trap. $EBCF$ - area of trap. $DACF$

$$\begin{aligned} &= \int_4^6 \left(\frac{5}{2}x - 9 \right) dx + \int_6^8 (-x + 12) dx - \int_4^8 \left(\frac{3}{4}x - 2 \right) dx \\ &= \frac{5}{2} \left[\frac{x^2}{2} \right]_4^6 - 9[x]_4^6 - \left[\frac{x^2}{2} \right]_6^8 + 12[x]_6^8 - \frac{3}{4} \left[\frac{x^2}{2} \right]_4^8 + 2[x]_4^8 \\ &= \frac{5}{4}(36 - 16) - 9(6 - 4) - \frac{1}{2}(64 - 36) + 12(8 - 6) - \frac{3}{8}(64 - 16) \\ &= \frac{5}{4} \times 20 - 18 - \frac{28}{2} + 24 - \frac{3}{8} \times 48 + 8 \\ &= 25 - 18 - 14 + 24 - 18 + 8 = 7 \text{ sq units.} \end{aligned}$$

Q.20. Using integration, find the area of the region bounded by the lines $4x - y + 5 = 0$, $x + y - 5 = 0$ and $x - 4y + 5 = 0$.

Ans.

We have given

$$4x - y + 5 = 0 \quad \dots(i)$$

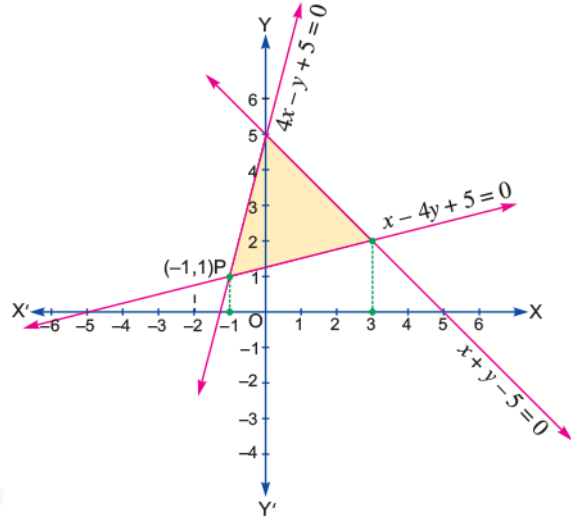
$$x + y - 5 = 0 \quad \dots(ii)$$

$$x - 4y + 5 = 0 \quad \dots(iii)$$

By solving equations (i) and (iii), we get $(-1, 1)$

and by solving (ii) and (iii), we get $(3, 2)$

\therefore Area of region bounded by the lines is given by:



$$\int_{-1}^0 \left\{ (4x + 5) - \left(\frac{x + 5}{4} \right) \right\} dx + \int_0^3 \left\{ (5 - x) - \left(\frac{x + 5}{4} \right) \right\} dx$$

$$= \int_{-1}^0 \left[\frac{15x}{4} + \frac{15}{4} \right] dx + \int_0^3 \left[\frac{15}{4} - \frac{5x}{4} \right] dx$$

$$= \left[\frac{15x^2}{8} + \frac{15x}{4} \right]_{-1}^0 + \left[\frac{15x}{4} - \frac{5x^2}{8} \right]_0^3$$

$$= 0 - \left(\frac{15}{8} - \frac{15}{4} \right) + \left(\frac{45}{4} - \frac{45}{8} \right) - 0$$

$$= \frac{15}{8} + \frac{45}{8} = \frac{15}{2} \text{ sq units.}$$

Q.21. Using integration, find the area of the following region:

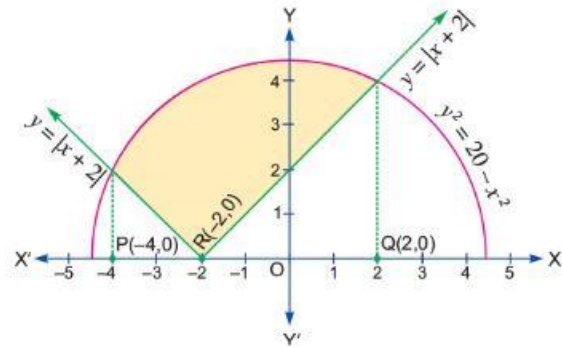
$$\left\{ (x, y) : |x + 2| \leq y \leq \sqrt{20 - x^2} \right\}$$

Ans.

Given region is $\{(x, y) : |x + 2| \leq y \leq \sqrt{20 - x^2}\}$

It consists of inequalities $y \geq |x + 2|$ and $y \leq \sqrt{20 - x^2}$

Plotting these inequalities, we obtain the adjoining shaded region.



Solving $y = x + 2$ and $y^2 = 20 - x^2$

$$\Rightarrow (x + 2)^2 = 20 - x^2$$

$$\Rightarrow 2x^2 + 4x - 16 = 0$$

$$\text{or } (x + 4)(x - 2) = 0 \Rightarrow x = -4, 2$$

$$\text{The required area} = \int_{-4}^2 \sqrt{20 - x^2} dx - \int_{-4}^{-2} -(x + 2) dx - \int_{-2}^2 (x + 2) dx$$

$$= \left[\frac{x}{2} \sqrt{20 - x^2} + \frac{20}{2} \sin^{-1} \frac{x}{\sqrt{20}} \right]_{-4}^2 + \left[\frac{x^2}{2} + 2x \right]_{-4}^{-2} - \left[\frac{x^2}{2} + 2x \right]_{-2}^2$$

$$= 4 + 10 \sin^{-1} \frac{1}{\sqrt{5}} + 4 + 10 \sin^{-1} \left(\frac{2}{\sqrt{5}} \right) + [2 - 4 - 8 + 8] - [2 + 4 - 2 + 4]$$

$$= 8 + 10 \left(\sin^{-1} \frac{1}{\sqrt{5}} + \sin^{-1} \frac{2}{\sqrt{5}} \right) - 2 - 8 = -2 + 10 \left(\sin^{-1} \frac{1}{\sqrt{5}} + \sin^{-1} \frac{2}{\sqrt{5}} \right)$$

$$= -2 + 10 \sin^{-1} \left[\frac{1}{\sqrt{5}} \sqrt{1 - \frac{4}{5}} + \frac{2}{\sqrt{5}} \sqrt{1 - \frac{1}{5}} \right] = -2 + 10 \sin^{-1} \left[\frac{1}{5} + \frac{4}{5} \right] = -2 + 10 \sin^{-1} 1$$

$$= -2 + 10 \frac{\pi}{2} = (5\pi - 2) \text{ sq units.}$$

Q.22. Using integration find the area of the triangular region whose sides have equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

Ans.

The given lines are

$$y = 2x + 1 \quad \dots(i)$$

$$y = 3x + 1 \quad \dots(ii)$$

$$x = 4 \quad \dots(iii)$$

For intersection point of (i) and (iii)

$$y = 2 \times 4 + 1 = 9$$

Coordinates of intersecting point of (i) and (iii) is (4, 9)

For intersection point of (ii) and (iii)

$$y = 3 \times 4 + 1 = 13$$

i.e., Coordinates of intersection point of (ii) and (iii) is (4, 13)

For intersection point of (i) and (ii)

$$2x + 1 = 3x + 1 \quad \Rightarrow x = 0$$

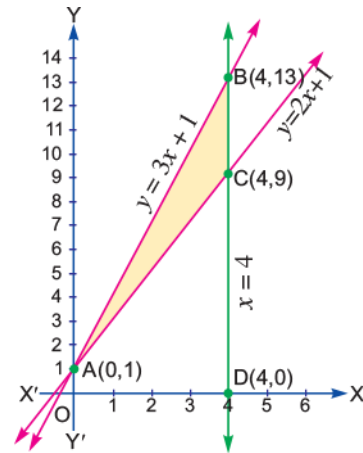
$$\therefore y = 1$$

i.e., Coordinates of intersection point of (i) and (ii) is (0, 1).

Shaded region is required triangular region.

\therefore Required Area = Area of trapezium OABD – Area of trapezium OACD

$$\begin{aligned} &= \int_0^4 (3x + 1) dx - \int_0^4 (2x + 1) dx = \left[\frac{3x^2}{2} + x \right]_0^4 - \left[\frac{2x^2}{2} + x \right]_0^4 \\ &= [(24 + 4) - 0] - [(16 + 4) - 0] = 28 - 20 = 8 \text{ sq units.} \end{aligned}$$

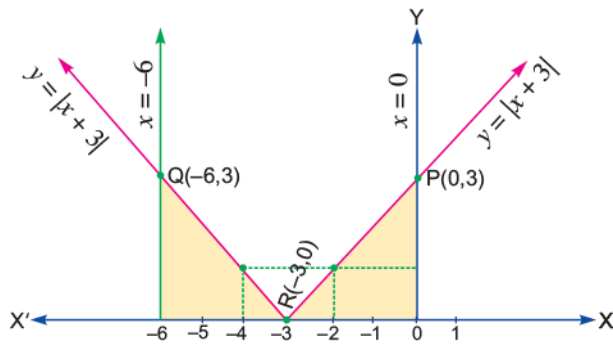


Q.23. Sketch the graph of $y = |x+3|$ and evaluate the area under the curve $y = |x+3|$ above x-axis and between $x = -6$ to $x = 0$.

Ans.

For graph of $y = |x+3|$

| | | | | | |
|----------|---|----|----|----|----|
| x | 0 | -3 | -6 | -2 | -4 |
| y | 3 | 0 | 3 | 1 | 1 |



Shaded region is the required region.

$$\text{Hence, Required area} = \int_{-6}^0 |x+3| dx$$

$$= \int_{-6}^{-3} |x+3| dx + \int_{-3}^0 |x+3| dx \quad [\text{By Property of definite integral}]$$

$$= \int_{-6}^{-3} |x+3| dx + \int_{-3}^0 |x+3| dx$$

$$= \int_{-6}^{-3} -(x+3) dx + \int_{-3}^0 (x+3) dx \quad \left[\begin{array}{l} x+3 \geq 0 \text{ if } -3 \leq x \leq 0 \\ x+3 \leq 0 \text{ if } -6 \leq x \leq -3 \end{array} \right]$$

$$= -\left[\frac{x^2}{2} + 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^0$$

$$= -\left[\left(\frac{9}{2} - 9\right) - \left(\frac{36}{2} - 18\right)\right] + \left[0 - \left(\frac{9}{2} - 9\right)\right]$$

$$= \frac{9}{2} + \frac{9}{2} = 9 \text{ sq units.}$$

Q.24. Using the method of integration, find the area of the region bounded by the lines:

$$2x + y = 4$$

$$3x - 2y = 6$$

$$x - 3y + 5 = 0$$

Ans.

Given lines are $2x + y = 4 \quad \dots(i)$

$$3x - 2y = 6 \quad \dots(ii)$$

$$x - 3y + 5 = 0 \quad \dots(iii)$$

For intersection point of (i) and (ii)

Multiplying (i) by 2 and adding with (ii), we get

$$4x + 2y = 8$$

$$\frac{3x - 2y - 6}{7x - 14} \Rightarrow x = 2 \text{ and } y = 0$$

Hence, intersection point of (i) and (ii) is (2, 0).

For intersection point of (i) and (iii)

Multiplying (i) by 3 and adding with (iii), we get

$$6x + 3y = 12$$

$$\frac{x - 3y - 5}{7x - 7} \Rightarrow x = 1 \text{ and } y = 2$$

Hence, intersection point of (i) and (iii) is (1, 2).

For intersection point of (ii) and (iii)

Multiplying (iii) by 3 and subtracting from (ii), we get

$$\begin{aligned} 3x - 2y &= 6 \\ \frac{-3x + 9y - 15}{7y - 21} &\Rightarrow y = 3 \text{ and } x = 4 \end{aligned}$$

Hence, intersection point of (ii) and (iii) is (4, 3).

With the help of intersecting points, required region ΔABC is plotted.

Shaded region is required region.

Required area = area of ΔABC

= Area of trap $ABED$ - Area of ΔADC - Area of ΔCBE

$$\begin{aligned} &= \int_1^4 \frac{x+5}{3} dx - \int_1^2 (4-2x) dx - \int_2^4 \frac{3x-6}{2} dx \\ &= \frac{1}{3} \left[\frac{x^2}{2} + 5x \right]_1^4 - [4x - x^2]_1^2 - \frac{1}{2} \left[\frac{3x^2}{2} - 6x \right]_2^4 \\ &= \frac{1}{3} \left\{ \left(\frac{16}{2} + 20 \right) - \left(\frac{1}{2} + 5 \right) \right\} - \left\{ (8-4) - (4-1) \right\} - \frac{1}{2} \left\{ \left(\frac{3 \times 16}{2} - 24 \right) - \left(\frac{3 \times 4}{2} - 12 \right) \right\} \\ &= \frac{1}{3} \left\{ 28 - \frac{11}{2} \right\} - \left\{ 4 - 3 \right\} - \frac{1}{2} \left\{ 0 + 6 \right\} = \frac{1}{3} \times \frac{45}{2} - 1 - 3 = \frac{7}{2} \text{ sq units} \end{aligned}$$

Q.25. Find the area of the region bounded by the parabola $y^2 = 2x$ and the line $x - y = 4$.

Ans.

Given curves are $y^2 = 2x$ (i)

and $x - y = 4$ (ii)

Obviously, curve (i) is right handed parabola having vertex at (0, 0) and axis along +ve direction of x-axis while curve (ii) is a straight line.

For intersection point of curve (i) and (ii)

$$(x - 4)^2 = 2x$$

$$\Rightarrow x^2 - 8x + 16 = 2x$$

$$\Rightarrow x^2 - 10x + 16 = 0$$

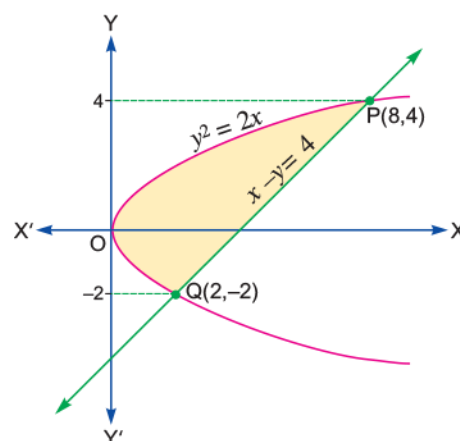
$$\Rightarrow x^2 - 8x - 2x + 16 = 0$$

$$\Rightarrow x(x - 8) - 2(x - 8) = 0$$

$$\Rightarrow (x - 8)(x - 2) = 0$$

$$\Rightarrow x = 2, 8$$

$$\Rightarrow y = -2, 4$$



Intersection points are $(2, -2), (8, 4)$

Therefore, required Area = Area of shaded region

$$= \int_{-2}^4 (y + 4) dy - \int_{-2}^4 \frac{y^2}{2} dy = \left[\frac{(y+4)^2}{2} \right]_{-2}^4 - \frac{1}{2} \left[\frac{y^3}{3} \right]_{-2}^4$$

$$= \frac{1}{2} \cdot [64 - 4] - \frac{1}{6} [64 + 8] = 30 - \frac{72}{6} = 18 \text{ sq units.}$$

Q.26. Using integration find the area of the region $\{(x, y) : x^2 + y^2 \leq 2ax, y^2 \geq ax, x, y \geq 0\}$.

Ans.

Given region R is

$$R = \{(x, y) : x^2 + y^2 \leq 2ax, y^2 \geq ax, x, y \geq 0\}.$$

$\Rightarrow R = R_1 \cap R_2 \cap R_3$, where

$$\Rightarrow R_1 = \{(x, y) : x^2 + y^2 \leq 2ax\}$$

$$R_2 = \{(x, y) : y^2 \geq ax\}$$

and $R_3 = \{(x, y) : x \geq 0, y \geq 0\}$

Obviously, $x^2 + y^2 = 2ax \Rightarrow (x - a)^2 + (y - 0)^2 = a^2$ is a circle having centre at $(a, 0)$ and radius $r = a$.

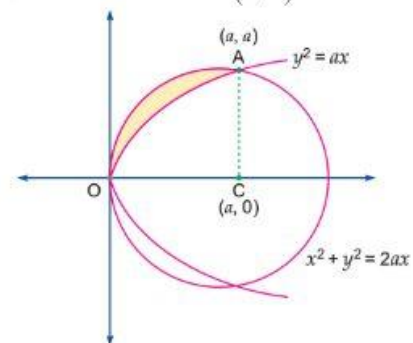
Therefore the region $R_1 \equiv \{(x, y) : x^2 + y^2 \leq 2ax\}$ is the region inside the circle with centre $(a, 0)$ and radius a .

Also $y^2 = ax$ is right handed parabola with vertex at origin.

So, region $R_2 \equiv \{(x, y) : y^2 \geq ax\}$ is the region outside parabola.

Also, $R_3 \equiv \{(x, y) : x \geq 0, y \geq 0\}$ is region in first quadrant.

Hence, $R = R_1 \cap R_2 \cap R_3$ is the shaded region shown above in figure.



Now for co-ordinate of A , we solve $y^2 = ax$ and

$x^2 + y^2 = 2ax$ as follows

$$x^2 + 2ax = 2ax \quad [\text{Putting } y^2 = ax]$$

$$x^2 - ax = 0$$

$$\Rightarrow x(x - a) = 0$$

$$\Rightarrow x = 0, a$$

For $x = a$, $y = a$

For $x = a$, $y = a$

Hence co-ordinate of A is (a, a)

$$\begin{aligned}
 \therefore \text{ Required area} &= \int_0^a \sqrt{2ax - x^2} dx - \int_0^a \sqrt{ax} dx \\
 &= \int_0^a \sqrt{-(x^2 - 2ax + a^2 - a^2)} dx - \sqrt{a} \int_0^a x^{1/2} dx \\
 &= \int_0^a \sqrt{-(x - a)^2 + a^2} dx - \sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_0^a \\
 &= \int_0^a \sqrt{a^2 - (x - a)^2} dx - \frac{2\sqrt{a}}{3} [a^{3/2} - 0] \\
 &= \left[\frac{1}{2} (x - a) \sqrt{a^2 - (x - a)^2} + \frac{1}{2} a^2 \sin^{-1} \left(\frac{x - a}{a} \right) \right]_0^a - \frac{2a^2}{3} \\
 &= \left[0 - \left\{ 0 + \frac{1}{2} a^2 \left(-\frac{\pi}{2} \right) \right\} \right] - \frac{2a^2}{3} \\
 &= \frac{\pi}{4} a^2 - \frac{2a^2}{3} = \left(\frac{\pi}{4} - \frac{2}{3} \right) a^2 \text{ sq unit.}
 \end{aligned}$$

Q.28. Find the area of the region $\{(x, y): x^2 + y^2 \leq 1 \leq x + y\}$.

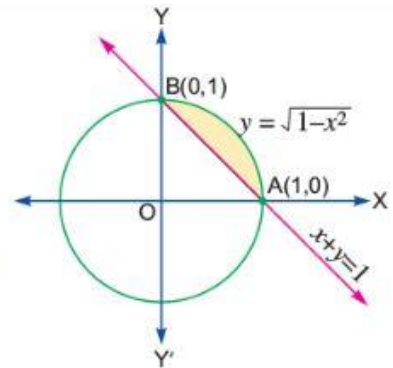
Ans.

The required area is the area between the circle

$$x^2 + y^2 = 1. \quad \dots(i)$$

and line $x + y = 1$...*(ii)*

Circle *(i)* has centre $(0, 0)$ and radius 1. Line *(ii)* meets x -axis at $A(1, 0)$ and y -axis at $B(0, 1)$. The circle *(i)* also passes through A and B . Hence, points of intersection of *(i)* and *(ii)* are $A(1, 0)$ and $B(0, 1)$



$$\begin{aligned}
 \text{Required area} &= \int_0^1 \sqrt{1 - x^2} dx - \int_0^1 (1 - x) dx \\
 &= \left[\frac{x\sqrt{1 - x^2}}{2} + \frac{1}{2} \sin^{-1} x \right]_0^1 - \left[x - \frac{x^2}{2} \right]_0^1 \\
 &= \frac{1}{2} \sin^{-1} (1) - \frac{1}{2} = \frac{1}{2} \left(\frac{\pi}{2} \right) - \frac{1}{2} = \left(\frac{\pi}{4} - \frac{1}{2} \right) \text{sq unit.}
 \end{aligned}$$

Q.28. Using integration find the area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

Ans.

Given circle is $x^2 + y^2 = 4$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 0 \quad [\text{By differentiating}]$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\text{Now, slope of tangent at } (1, \sqrt{3}) = \left. \frac{dy}{dx} \right|_{(1, \sqrt{3})} = -\frac{1}{\sqrt{3}}$$

$$\therefore \text{Slope of normal at } (1, \sqrt{3}) = \sqrt{3}$$

Therefore, equation of tangent is

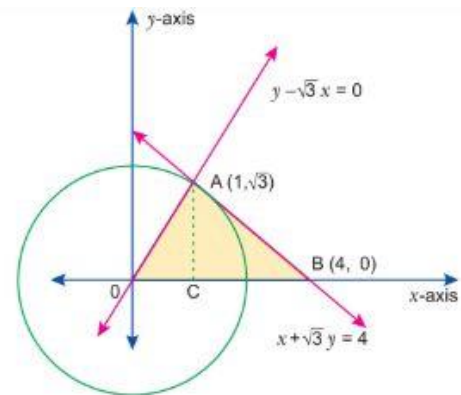
$$\frac{y - \sqrt{3}}{x - 1} = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow x + \sqrt{3}y = 4 \dots (i)$$

Again, equation of normal is

$$\frac{y - \sqrt{3}}{x - 1} = \sqrt{3}$$

$$\Rightarrow y - \sqrt{3}x = 0 \dots (ii)$$



To draw the graph of the triangle formed by the lines x -axis, (i) and (ii), we find the intersecting points of these three lines which give vertices of required triangle. Let O, A, B be the intersecting points of these lines.

Obviously, the coordinate of O, A, B are $(0, 0), (1, \sqrt{3})$ and $(4, 0)$ respectively.

Required area = area of triangle OAB = area of region OAC + area of region CAB

$$= \int_0^1 y \, dx + \int_1^4 y \, dx \quad [\text{Where in 1st integrand } y = \sqrt{3}x \text{ and in 2nd } y = \frac{4-x}{\sqrt{3}}]$$

$$= \int_0^1 \sqrt{3}x \, dx + \int_1^4 \frac{4-x}{\sqrt{3}} \, dx = \sqrt{3} \left[\frac{x^2}{2} \right]_0^1 - \frac{1}{\sqrt{3}} \left[\frac{(4-x)^2}{2} \right]_1^4$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}} \left[0 - \frac{9}{2} \right]$$

$$= \frac{\sqrt{3}}{2} + \frac{9}{2\sqrt{3}} = \frac{12}{2\sqrt{3}} = 2\sqrt{3} \text{ sq units.}$$

Q.29. Find the area of the region $\{(x, y) : x \leq y \leq |x|\}$.

Ans.

Given curves are

$$x^2 + y^2 = 16 \quad \dots(i)$$

$$y^2 = 6x \quad \dots(ii)$$

Obviously curve (i) is a circle having centre at $(0, 0)$ and radius 4 unit.

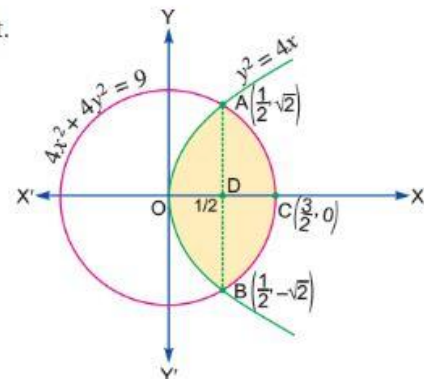
While curve (ii) is right handed parabola having vertex at $(0, 0)$ and axis along +ve direction of x -axis.

Required part in the shaded region.

Now, for intersection point of curve (i) & (ii)

$$x^2 + 6x = 16$$

$$\Rightarrow x^2 + 6x - 16 = 0$$



$$\begin{aligned}
\Rightarrow x^2 + 8x - 2x - 16 &= 0 \\
\Rightarrow x(x + 8) - 2(x + 8) &= 0 \\
\Rightarrow (x + 8)(x - 2) &= 0 \\
\Rightarrow x = -8 \text{ or } 2 \quad [\because x = -8 \text{ is not possible as } y^2 \text{ is +ve}] \\
\therefore x &= 2 \\
\therefore y &= \pm \sqrt{12} = \pm 2\sqrt{3}
\end{aligned}$$

Hence, coordinate of B is $(2, 2\sqrt{3})$.

Since, shaded region *i.e.*, required part is symmetrical about x -axis.

Therefore, area of required part

$$\begin{aligned}
&= 2 [\text{Area } OBEDO] \\
&= 2 [\text{Area } DEBFD - \text{Area } OBFO] \\
&= 2 \left[\int_{-4}^2 \sqrt{16 - x^2} \, dx - \int_0^2 \sqrt{6x} \, dx \right] \\
&= 2 \left[\frac{x}{2} \sqrt{16 - x^2} + \frac{16}{2} \sin^{-1} \frac{x}{4} \right]_{-4}^2 - 2\sqrt{6} \cdot \frac{2}{3} [x^{3/2}]_0^2 \\
&= \left[\left(2 \times \sqrt{12} + 16 \sin^{-1} \frac{1}{2} \right) - (-4) \times 0 + 16 \sin^{-1}(-1) \right] - \frac{4\sqrt{6}}{3} [2\sqrt{2} - 0] \\
&= \left(\frac{8}{3} + 8 \right) \pi + \left(4 - \frac{16}{3} \right) \sqrt{3} \\
&= \left(\frac{32}{3} \pi - \frac{4}{3} \sqrt{3} \right) \text{ sq units.}
\end{aligned}$$

Long Answer Question-II (OIQ)

[6 marks]

Q.1. Using integration, find the area of the region bounded between the line $x = 4$ and the parabola $y^2 = 16x$.

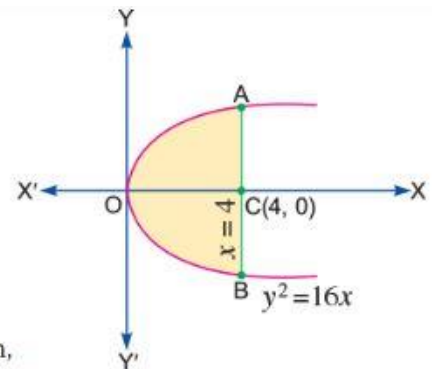
Ans.

Given parabola and line are

$$y^2 = 16x \quad \dots(i)$$

$$x = 4 \quad \dots(ii)$$

Obviously, given parabola (i) is right handed parabola with vertex at (0, 0) and axis along +ve direction of x-axis while line (ii) is a straight line parallel to y-axis meeting x-axis at x = 4. Shaded region is required region, which is symmetric along x-axis.



\therefore Required area A is given by

$$A = 2(\text{area } OCAO)$$

$$\Rightarrow A = 2 \int_0^4 y \, dx = 2 \int_0^4 \sqrt{16x} \, dx = 8 \int_0^4 \sqrt{x} \, dx \quad [\because y^2 = 16x \quad \therefore y = \sqrt{16x}]$$

$$\therefore A = 8 \left[\frac{x^{3/2}}{3/2} \right]_0^4 = \frac{16}{3} (4^{3/2} - 0^{3/2}) = \frac{16}{3} \times 8 = \frac{128}{3} \text{ sq units.}$$

Q.2. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.

Ans.

Given curve is

$$y^2 = 2y - x \quad \dots(i)$$

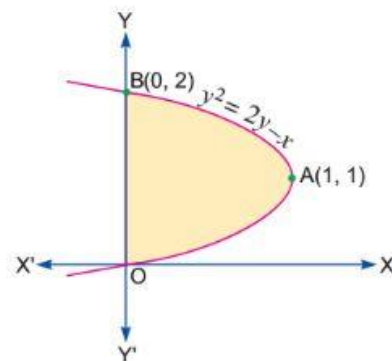
$$y^2 - 2y = -x$$

$$\Rightarrow y^2 - 2y + 1 = -x + 1$$

$$\Rightarrow (y - 1)^2 = -(x - 1)$$

Obviously, given curve is left handed parabola having vertex at (1, 1).

Putting x = 0 in (i), we get



$$y^2 = 2y \Rightarrow y(y - 2) = 0 \Rightarrow y = 0, 2$$

i.e., given parabola meets y axis at $y = 0$ and $y = 2$, so shaded region is required region.

\therefore Required area = Area of region $OABO$

$$= \int_0^2 (2y - y^2) dy = \left[y^2 - \frac{y^3}{3} \right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ sq units.}$$

Q.3. Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Ans.

The point of intersection of these two parabolas are $O(0, 0)$ and $A(1, 1)$ as shown in the fig.

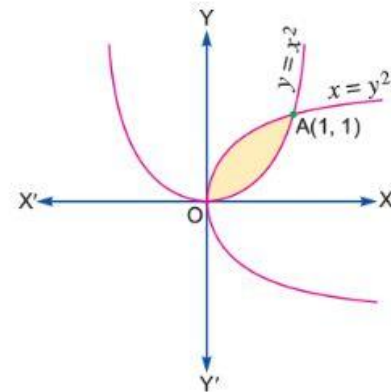
Here, we can set $y^2 = x$ or $y = \sqrt{x} = f(x)$ (because region lies above x -axis) and $y = x^2 = g(x)$ where, $f(x) \geq g(x)$ in $[0, 1]$.

Therefore, the required area of the shaded region

$$= \int_0^1 [f(x) - g(x)] dx$$

$$= \int_0^1 [\sqrt{x} - x^2] dx = \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq unit}$$



Q.4. Using integration, find the area bounded by the tangent to the curve $4y = x^2$ at the point $(2, 1)$ and the lines whose equations are $x = 2y$ and $x = 3y - 3$.

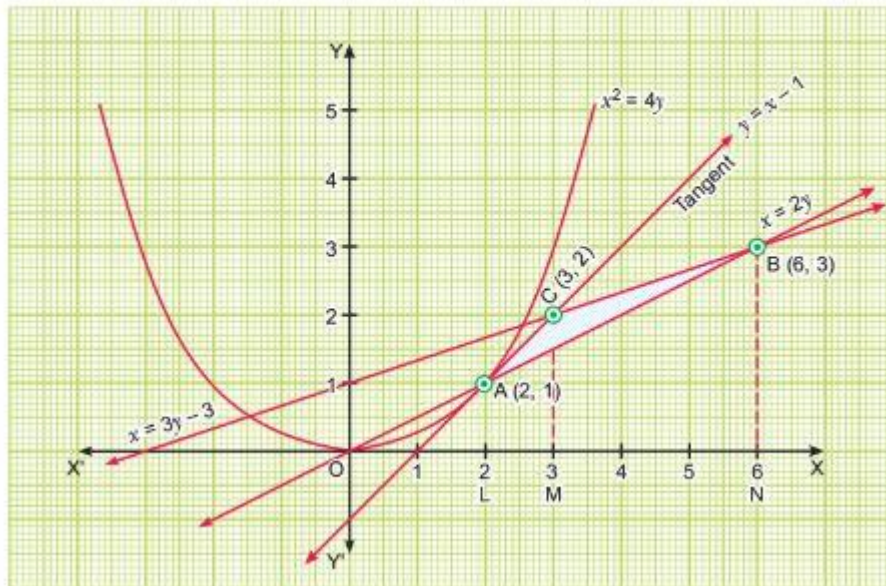
Ans.

Obviously $4y = x^2$ is upward parabola having vertex at origin.

Now $4y = x^2$

$$\Rightarrow 4 \frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{1}{2} x \Rightarrow \left. \frac{dy}{dx} \right|_{(2,1)} = \frac{1}{2} \times 2 = 1$$

⇒ Slope of tangent at (2, 1) to given curve $4y = x^2$ is 1.



$$\text{Equation of tangent} = \frac{y-1}{x-2} = 1$$

$$\Rightarrow y - 1 = x - 2 \quad \Rightarrow \quad y = x - 1$$

Now, for graph of $x = 2y$

| | | |
|----------|---|---|
| x | 0 | 2 |
| y | 0 | 1 |

Also for graph of $x = 3y - 3$

| | | |
|----------|---|---|
| x | 0 | 3 |
| y | 1 | 2 |

After plotting the graph, we get shaded region ABC as required region, area of which is to be calculated.

After solving the respective equation, we get

Coordinate of $A \equiv (2, 1)$; $B \equiv (6, 3)$; $C \equiv (3, 2)$

Now, the required area = area of shaded region ABC

$$= \text{ar}(\text{region } ALMC) + \text{ar}(\text{region } CMNB) - \text{ar}(\text{region } ALNB)$$

$$= \text{ar}(\text{region } ALMC) + \text{ar}(\text{region } CMNB) - \text{ar}(\text{region } ALNB)$$

$$= \int_2^3 (x-1)dx + \int_3^6 \frac{x+3}{3}dx - \int_2^6 \frac{x}{2}dx = \left[\frac{x^2}{2} - x \right]_2^3 + \frac{1}{3} \left[\frac{x^2}{2} + 3x \right]_3^6 - \frac{1}{2} \left[\frac{x^2}{2} \right]_2^6$$

$$= \left[\left(\frac{9}{2} - 3 \right) - \left(\frac{4}{2} - 2 \right) \right] + \frac{1}{3} \left[\left(\frac{36}{2} + 18 \right) - \left(\frac{9}{2} + 9 \right) \right] - \frac{1}{4} (36 - 4)$$

$$= \frac{3}{2} + \frac{1}{3} \left(36 - \frac{27}{2} \right) - 8 = \frac{3}{2} + \frac{45}{6} - 8 = 1 \text{ square unit.}$$

Q.5. Sketch the graph $y = |x + 1|$. Evaluate $\int_{-3}^1 |x + 1| dx$. What does this value represent on the graph?

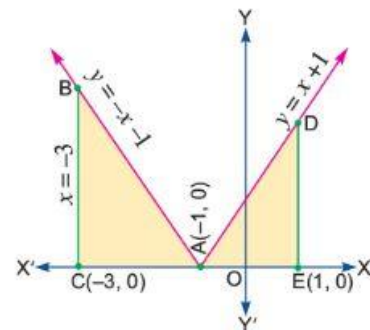
$$\text{We have, } y = |x + 1| = \begin{cases} x + 1, & \text{if } x + 1 \geq 0 \text{ i.e., } x \geq -1 \\ -(x + 1), & \text{if } x + 1 < 0 \text{ i.e., } x < -1 \end{cases}$$

So, we have $y = x + 1$ for $x \geq -1$ and $y = -x - 1$ for $x < -1$. Clearly, $y = x + 1$ is a straight line cutting x and y -axes at $(-1, 0)$ and $(0, 1)$ respectively. So, $y = x + 1, x \geq -1$ represents that portion of the line which lies on the right side of $x = -1$. Similarly, $y = -x - 1, x < -1$ represents that part of the line $y = -x - 1$ which is on the left side of $x = -1$. A rough sketch of $y = |x + 1|$ is shown in fig.

$$\text{Now, } \int_{-3}^1 |x + 1| dx = \int_{-3}^{-1} -(x + 1) dx + \int_{-1}^1 (x + 1) dx$$

$$= \left[\frac{(x+1)^2}{2} \right]_{-3}^{-1} + \left[\frac{(x+1)^2}{2} \right]_{-1}^1 = \left[0 - \frac{4}{2} \right] + \left[\frac{4}{2} - 0 \right] = 4 \text{ sq units}$$

This value represents the area of the shaded portion shown in figure.



Q.6. Find the area of the region $\{(x, y) : x^2 \leq y \leq x\}$

Ans.

We have, $y = x^2$... (i)

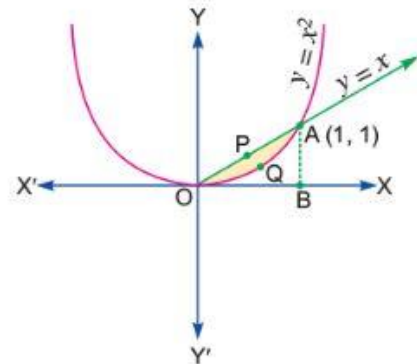
and $y = x$... (ii)

We know that $y = x^2$ is an upward parabola and the line $y = x$ is passing through origin.

Now, on solving (i) and (ii), we get

$$x^2 = x \Rightarrow x(x - 1)$$

$$x = 0 \text{ or } 1$$



From (ii), $x = 0 \Rightarrow y = 0$ and $x = 1 \Rightarrow y = 1$

So, the point of intersection of (i) and (ii) are $O(0, 0)$ and $A(1, 1)$.

Draw $AB \perp OX$

Required area = Shaded area shown in figure

= area $OPABO$ - area $OQABO$

$$= \int_0^1 x \, dx - \int_0^1 x^2 \, dx$$

$$= \left[\frac{x^2}{2} \right]_0^1 - \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq unit.}$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq unit.}$$

Hence, the required area is $\frac{1}{6}$ sq unit.