Very Short Answer Questions (PYQ)

[1 Mark]

Q.1. What is the degree of the following differential equation?

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x?$$

Ans. Degree of differential equation is 1 because power of highest order derivative $\frac{d^2y}{dx^2}$ is one.

Q.2. Find the order and degree of differential equation:

$$\frac{d^4y}{dx^2} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$$

Ans.

Order is 4 but degree is not defined because given differential equation cannot be written in the form of polynomial in differential co-efficient.

Q.3. Find the differential equation representing the curve $y = cx + c^2$.

Ans.

Given $y = cx + c^2$

$\Rightarrow \quad \frac{dy}{dx} = c + 0$ [Differentiating with respect to x]

$\Rightarrow \quad \frac{dy}{dx} = c$

Q.4. Find the differential equation representing the curve $y = e^{-x} + ax + b$, where *a* and *b* are arbitrary constants.

Ans.

Given curve is

$$y = e^{-x} + ax + b$$

Differentiating with respect to *x*, we get

$$\frac{dy}{dx} = -e^{-x} + a$$

Differentiating again with respect to *x*, we get

$$\frac{d^2y}{dx^2} = e^{-x}$$

Q.5. Find the differential equation representing the family of curves $v = \frac{A}{r} + B$, where *A* and *B* are arbitrary constants.

Ans.

Given family of curve is

$$v = \frac{A}{r} + B$$

Differentiating with respect to *r*, we get

$$\frac{dv}{dr} = \frac{-A}{r^2}$$

Again differentiating with respect to *r*, we get

$$\frac{d^2v}{dr^2} = \frac{2A}{r^3}$$

$$\Rightarrow \quad \frac{d^2v}{dr^2} = \frac{2}{r} \cdot \frac{A}{r^2}$$

$$\Rightarrow \quad \frac{d^2v}{dr^2} = \frac{2}{r} \cdot \left(-\frac{dv}{dr}\right)$$

$$\Rightarrow \quad r\frac{d^2v}{dr^2} = -2\frac{dv}{dr}$$

$$\Rightarrow \quad r\frac{d^2v}{dr^2} + 2\frac{dv}{dr} = 0$$

Q.6. Write the sum of the order and degree of the following differential equation:

$$\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\}=0$$

Ans.

Given differential equation is

$$\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^3\right]=0$$

$$\Rightarrow \quad 3\left(\frac{dy}{dx}\right)^2.\frac{d^2y}{dx^2}=0$$

i.e., order = 2, degree = 1

$\therefore$ Required sum = 2 + 1 = 3.

Very Short Answer Questions (OIQ)

[1 Mark]

Q.1. Write the order and degree of each of the following differential equations:

i. $\frac{d^3y}{dx^3}+2\left(\frac{d^2y}{dx^2}\right)-\frac{dy}{dx}+y=0$

ii. $\left(\frac{d^2y}{dx^2}\right)^2+x^2\left(\frac{dy}{dx}\right)^3=0$

Ans.

i. Order = 3, degree = 1
ii. Order = 2, degree = 2

Q.2. Write the integrating factor of

$$\frac{dy}{dx}-\left(\frac{1}{1+x}\right)y=(1+x)e^x$$

Ans.

$$\text{IF}=e^{\int-\left(\frac{1}{1+x}\right)dx}=e^{-\log\ (1+x)}=e^{\log\left|(1+x)^{-1}\right|}=\frac{1}{1+x}$$

Q.3. Write the integrating factor of

$$(x \log x)\frac{dy}{dx} + y = 2\log x$$

Ans.

From the given differential equation, we have

$$\frac{dy}{dx} + \frac{1}{x\log x}\cdot y = \frac{2}{x}$$

So, IF $= e^{\int \frac{1}{x\log x}dx} = e^{\log|(\log x)|} = \log x$

Q.4. Write the integrating factor of

Ans.

$$\frac{dx}{dy} + (\tan y)x = \sec^2 y$$

Short Answer Questions

[2 Mark]

Q.1. Write the integrating factor of the following differential equation:

$$(1+y^2)+(2\,xy-\cot y)\frac{dy}{dx}=0$$

Ans.

$$(1+y^2)+(2\,xy-\cot y)\frac{dy}{dx}=0$$

$$\Rightarrow\quad(2\,xy-\cot y)\frac{dy}{dx}=-(1+y^2)$$

$$\Rightarrow\quad\frac{dy}{dx}=-\frac{1+y^2}{2\,xy-\cot y}$$

$$\Rightarrow\quad\frac{dx}{dy}=-\frac{(2\,xy-\cot y)}{1+y^2}$$

$$\Rightarrow\quad\frac{dx}{dy}+\frac{2y}{1+y^2}.x=\frac{\cot y}{1+y^2}$$

It is in the form $\frac{dx}{dy}+Px=Q,$ where P and Q are function of y

$$\Rightarrow\quad \mathrm{IF}=e^{\int p\ dy}=e^{\int\frac{2y}{1+y^2}dy}=e^{\log|1+y^2|}=1+y^2$$

Q.2. Write the general solution of the differential equation $\frac{dy}{dx}=\frac{y}{x}$

Ans.

We have, $\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \quad \frac{dy}{y} = \frac{dx}{x}$

Integrating both sides, we get

$$\log|y| = \log|x| + \log|C|$$

$\Rightarrow \quad |y| = |xC|$

$\Rightarrow \quad y = Cx$

Q.3. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

Ans.

Given $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$\Rightarrow \quad \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Integrating both sides we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$\Rightarrow \quad \tan^{-1} y = \tan^{-1} x + C$

Q.4. Find the general solution of differential equation.

$\frac{dy}{dx} = \frac{1 - \cos x}{1+\cos x}$

Ans.

Given $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \tan^2\frac{x}{2}$$

$$\Rightarrow \quad dy = \left(\sec^2\frac{x}{2} - 1\right) dx$$

Integrating both sides

$$\Rightarrow \quad \int dy = \int \sec^2 x/2 dx - \int dx$$

$$\Rightarrow \quad y = 2\tan\frac{x}{2} - x + C$$

Q.5. Solve the differential equation

$$\frac{dy}{dx} = \left(1 + x^2\right)\left(1 + y^2\right)$$

Ans.

Given $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

$$\Rightarrow \quad \frac{dy}{1+y^2} = (1 + x^2)\ dx$$

$$\Rightarrow \quad \int \frac{dy}{1+y^2} = \int (1 + x^2)\ dx$$

$$\Rightarrow \quad \tan^{-1} y = x + \frac{x^3}{3} + C$$

Q.6. Solve the differential equation.

$\mathbf{y} \log \mathbf{y}\ \mathbf{dx} - \mathbf{xdy} = 0$

Ans.

Given $y \log y\ dx - xdy = 0$]

$\Rightarrow \quad y \log y\ dx = xdy$

$\Rightarrow \quad \frac{dx}{x} = \frac{dy}{y \log y}$

$\Rightarrow \quad \int \frac{dy}{y \log y} = \int \frac{dx}{x}$

Let $\log y = t$

$\Rightarrow \quad \frac{1}{y} dy = dt$

$\Rightarrow \quad \int \frac{dt}{t} = \log x + C_1$

$\Rightarrow \quad \log t = \log x + \log C \qquad [\text{where } C_1 = \log C]$

$\Rightarrow \quad t = Cx$

$\Rightarrow \quad \log y = Cx$

$\Rightarrow \quad y = e^{Cx}$

Q.7. Solve differential equation $\frac{dx}{x} + \frac{dy}{y} = 0$.

Ans.

Given $\frac{dx}{x} + \frac{dy}{y} = 0$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log y = -\log x + \log C$$

$$\Rightarrow \quad \log y + \log x = \log C$$

$$\Rightarrow \quad \log xy = \log C$$

$$\Rightarrow \quad xy = C$$

Long Answer Questions-I (PYQ)

[4 Marks]

Q.1. Find the particular solution of the differential equation $log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$ when $x = 0$.

Ans.

Given differential equation is $\log \frac{dy}{dx} = 3x + 4y$

$\Rightarrow \quad \frac{dy}{dx} = e^{3x+4y}$

$\Rightarrow \quad \frac{dy}{dx} = e^{3x} . e^{4y}$

$\Rightarrow \quad \frac{dy}{e^{4y}} = e^{3x} . dx$

$\Rightarrow \quad e^{-4y} dy = e^{3x} dx$

Integrating both sides, we get

$$\int e^{-4y} dy = \int e^{3x} dx$$

$\Rightarrow \quad \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C_1$

$\Rightarrow \quad -3e^{-4y} = 4e^{3x} + 12C_1$

$\Rightarrow \quad 4e^{3x} + 3e^{-4y} = -12C_1$

$\Rightarrow \quad 4e^{3x} + 3e^{-4y} = C \qquad \ldots(i)$

It is general solution.

Now for particular solution we put $x = 0$ and $y = 0$ in (i)

$$4 + 3 = C \quad \Rightarrow \quad C = 7.$$

Putting $C = 7$ in (i), we get

$$4e^{3x} + 3e^{-4y} = 7$$

It is required particular solution.

Q.2. Solve the following differential equation:

$$2x^2 \frac{\mathrm{dy}}{\mathrm{dx}} - 2\mathrm{xy} + y^2 = 0$$

Ans.

Given $2x^2 \frac{\mathrm{dy}}{\mathrm{dx}} - 2\mathrm{xy} + y^2 = 0$

$$\Rightarrow \quad 2x^2 \frac{\mathrm{dy}}{\mathrm{dx}} = 2\mathrm{xy} - y^2$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{2\mathrm{xy} - y^2}{2x^2} \qquad \ldots(i)$$

It is homogeneous differential equation.

Let $y = \mathrm{vx}$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Equation (i) becomes

$$v + x\frac{dv}{dx} = \frac{2x.vx - v^2x^2}{2x^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{2x^2\left(v - \frac{v^2}{2}\right)}{2x^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = v - \frac{v^2}{2} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{v^2}{2}$$

$$\Rightarrow \quad \frac{dx}{x} = -\frac{2\,dv}{v^2}$$

Integrating both sides, we get

$$\Rightarrow \quad \int \frac{dx}{x} = -2\int \frac{dv}{v^2}$$

$$\Rightarrow \quad \log|x| + C = -2\frac{v^{-2+1}}{-2+1}$$

$$\Rightarrow \quad \log|x| + C = 2\,.\,\frac{1}{v}$$

Putting $v = \frac{y}{x}$, we get

$$\log|x| + C = \frac{2x}{y}$$

Q.3. Find the particular solution of the following differential equation:

$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, **given that** $y = 1$ **when** $x = 0$

Ans.

Given $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x^2) + y^2(1 + x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

$$\Rightarrow \quad (1 + x^2)dx = \frac{dy}{(1+y^2)}$$

Integrating both sides, we get

$$\int (1 + x^2)dx = \int \frac{dy}{(1+y^2)}$$

$$\Rightarrow \quad \int dx + \int x^2 dx = \int \frac{dy}{(1+y^2)}$$

$$\Rightarrow \quad x + \frac{x^3}{3} + C = \tan^{-1} y$$

Putting $y = 1$ and $x = 0$, we get

$$\tan^{-1}(1) = 0 + 0 + C$$

$$\Rightarrow \quad C = \tan^{-1}(1) = \frac{\pi}{4}$$

Therefore, required particular solution is

$$\tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$$

Q.4. Find the particular solution of the differential equation:

$x(x^2 - 1)\frac{dy}{dx} = 1;\ y = 0;$ when $x = 2$

Ans.

Given differential equation is,

$$x(x^2-1)\frac{dy}{dx}=1$$

$$\Rightarrow \quad dy=\frac{dx}{x\,(x^2-1)}$$

$$\Rightarrow \quad dy=\frac{dx}{x\,(x-1)\,(x+1)}$$

Integrating both sides, we get,

$$\int dy=\int\frac{dx}{x\,(x-1)\,(x+1)}$$

$$\Rightarrow \quad y=\int\frac{dx}{x(x-1)\,(x+1)} \qquad ...(i)$$

Let $\frac{1}{x(x-1)(x+1)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$

$$\Rightarrow \quad \frac{1}{x(x-1)(x+1)}=\frac{A(x-1)\,(x+1)+Bx\,(x+1)+Cx\,(x-1)}{x(x-1)(x+1)}$$

$$\Rightarrow \quad 1=A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$$

Putting $x=1$, we get $1=0+B.\,1.\,2+0 \Rightarrow B=\frac{1}{2}$

Putting $x=-1$, we get $1=0+0+C.(-1).(-2) \Rightarrow C=\frac{1}{2}$

Putting $x=0$, we get $1=A\,(-1).1 \Rightarrow A=-1$

Hence, $\frac{1}{x(x-1)(x+1)}=\frac{-1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$

From (i) $y = \int \left(-\frac{1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)} \right) \mathrm{dx}$

$\Rightarrow \quad y = -\int \frac{\mathrm{dx}}{x} + \frac{1}{2} \int \frac{\mathrm{dx}}{x-1} + \frac{1}{2} \int \frac{\mathrm{dx}}{x+1}$

$\Rightarrow \quad y = -\log x + \frac{1}{2} \log |x-1| + \frac{1}{2} \log |x+1| + \log C_1$

$\Rightarrow \quad 2y = 2 \log \frac{1}{x} + \log |x^2 - 1| + 2 \log C_1$

$\Rightarrow \quad 2y = \log \left| \frac{x^2-1}{x^2} \right| + \log C_1^2 \qquad ...(ii)$

When $x = 2,\ y = 0$

$\Rightarrow \quad 0 = \log \left| \frac{4-1}{4} \right| + \log C_1^2$

$\Rightarrow \quad \log C_1^2 = -\log \frac{3}{4}$

Putting $\log C_1^2 = -\log \frac{3}{4}$ in (ii) we get

$2y = \log \left| \frac{x^2-1}{x^2} \right| - \log \frac{3}{4}$

$\Rightarrow \quad y = \frac{1}{2} \log \left| \frac{x^2-1}{x^2} \right| - \frac{1}{2} \log \frac{3}{4}$

Q.5. Solve the differential equation $(1 + x^2))\frac{\mathrm{dy}}{\mathrm{dx}} + y = e^{\tan^{1} x}$.

Ans.

Given differential equation is

$$(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{1}{1+x^2}y=\frac{e^{\tan^{-1}x}}{1+x^2} \qquad \ldots(i)$$

Equation (i) is of the form

$$\frac{dy}{dx}+Py=Q, \text{ where } P=\frac{1}{1+x^2}, Q=\frac{e^{\tan^{-1}x}}{1+x^2}$$

$$\therefore \quad IF=e^{\int P dx}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$$

Therefore, general solution of required differential equation is

$$y.e^{\tan^{-1}x}=\int e^{\tan^{-1}x}.\frac{e^{\tan^{-1}x}}{1+x^2}dx+C$$

$$\Rightarrow \quad y.e^{\tan^{-1}x}=\int\frac{e^{2\tan^{-1}x}}{1+x^2}dx+C \qquad \ldots(ii)$$

Let $\tan^{-1}x=z \quad \Rightarrow \quad \frac{1}{1+x^2}dx=dz$

(ii) becomes

$$y.e^{\tan^{-1}x} = \int e^{2z}\text{dz} + C$$

$$\Rightarrow \quad y.e^{\tan^{-1}x} = \frac{e^{2z}}{2} + C$$

$$\Rightarrow \quad y.e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + C \qquad \left[\text{Putting } z = \tan^{-1}x\right]$$

$$\Rightarrow \quad y = \frac{e^{\tan^{-1}x}}{2} + C.e^{-\tan^{-1}x} \qquad \left[\text{Dividing both sides by } e^{\tan^{-1}x}\right]$$

It is required solution.

Q.6. Find the particular solution of the differential equation $e^{x}\sqrt{1-y^2}\text{dx} + \frac{y}{x}\text{dy} = 0$ **given that *y*= 1 when *x* = 0.**

Ans.

We have, $e^{x}\sqrt{1-y^2}\text{dx} + \frac{y}{x}\text{dy} = 0$

$$\Rightarrow \quad e^{x}\sqrt{1-y^2}\text{dx} = -\frac{y}{x}\text{dy}$$

$$\Rightarrow \quad \text{x}e^{x}\,\text{dx} = -\frac{y}{\sqrt{1-y^2}}\text{dy}$$

$$\Rightarrow \quad \int \underset{I}{x}\,\underset{\text{II}}{e^{x}}\,\text{dx} = -\int \frac{y}{\sqrt{1-y^2}}\text{dy}$$

$$\Rightarrow \quad \text{x}e^{x} - \int e^{x}\text{dx} = \frac{1}{2}\int \frac{\text{dt}}{\sqrt{t}}, \text{ where } t = 1 - y^2 \qquad \text{(Using ILATE on LHS)}$$

$$\Rightarrow \quad \mathrm{x}e^{x} - e^{x} = \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$\Rightarrow \quad \mathrm{x}e^{x} - e^{x} = \sqrt{t} + C$$

$$\Rightarrow \quad \mathrm{x}e^{x} - e^{x} = \sqrt{1-y^2} + C, \text{ is the required solution.}$$

Putting $y = 1$ and $x = 0$, we get

$$0e^0 - e^0 = \sqrt{1-1^2} + C$$

$$\Rightarrow \quad C = -1$$

Therefore, required particular solution is $\mathrm{x}e^{x} - e^{x} = \sqrt{1-y^2} - 1.$

Q.7. Solve: $x\frac{\mathrm{dy}}{\mathrm{dx}} = y - x\tan\left(\frac{y}{x}\right)$

Ans.

We have, $\quad x\frac{\mathrm{dy}}{\mathrm{dx}} = y - x\tan\left(\frac{y}{x}\right)$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \qquad \ldots(i)$$

Clearly, the given differential equation is homogeneous.

Putting $y = \mathrm{vx}$ and $\frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dv}}{\mathrm{dx}}$ in (i), we get

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = v - \tan v$$

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = v - \tan v$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \tan v$$

$$\Rightarrow \quad \cot v\,\mathrm{dv} = \frac{-\mathrm{dx}}{x}, \text{ if } x \neq 0 \qquad \text{[By separating the variables]}$$

Integrating both sides, we get

$$\int \cot v\,\mathrm{dv} = -\int \frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \log|\sin v| = -\log|x| + \log C$$

$$\Rightarrow \quad \log|\sin v| = \log C$$

$$\Rightarrow \quad \left|x \sin \frac{y}{x}\right| = |C|$$

Hence, $x \sin \frac{y}{x} = C$ is the required solution.

Q.8. Solve the differential equation:

$(\tan^{-1} y - x)\,dy = (1 + y^2)\,dx$

OR

Find the particular solution of the differential equation $(\tan^{-1} y - x)dy = (1 + y^2)dx$**, given that** $x = 1$ **when** $y = 0$**.**

Ans.

The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \qquad ...(i)$$

Now, (i) is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$,

where, $P = \frac{1}{1+y^2}$ and $Q = \frac{\tan^{-1} y}{1+y^2}$

Therefore, $IF = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$

Thus, the solution of the given differential equation is

$$xe^{\tan^{-1} y} = \int \left(\frac{\tan^{-1} y}{1+y^2}\right) e^{\tan^{-1} y} dy + C \qquad ...(ii)$$

Let $I = \int \left(\frac{\tan^{-1} y}{1+y^2}\right) e^{\tan^{-1} y}\, dy$

Substituting $\tan^{-1} y = t$ so that $\left(\frac{1}{1+y^2}\right) dy = dt$, we get

$$I = \int t e^t\, dt = t e^t - \int 1 . e^t\, dt = t e^t - e^t \equiv e^t (t-1)$$

or $\quad I = e^{\tan^{-1} y} (\tan^{-1} y - 1)$

Substituting the value of I in equation (ii), we get

$$x . e^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + C$$

or $\qquad x = (\tan^{-1} y - 1) + 2 e^{-\tan^{-1} y}$

Which is the general solution of the given differential equation.

OR

For general solution same as above.

General solution is $x = (\tan^{-1} y - 1) + Ce^{-\tan^{-1} y}$

For particular solution putting $x = 1$, $y = 0$, we get

$$1 = (\tan^{-1} 0 - 1) + C.e^{-\tan^{-1} 0}$$
$$1 = -1 + C \quad \Rightarrow \quad C = 2$$

Therefore required particular solution is

$$x = (\tan^{-1} y - 1) + e^{-\tan^{-1} y}$$

Q.9. Solve the following differential equation:

$$\left[\frac{e^{2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right]\frac{dx}{dy} = 1, x \neq 0$$

Ans.

Given $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1,\ x \neq 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{\sqrt{x}} \cdot y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is in the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{\sqrt{x}}, Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \quad IF = e^{\int P\ dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{\int x^{-\frac{1}{2}}\ dx} = e^{\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}} = e^{2\sqrt{x}}$$

Therefore general solution is

$$y \, . e^{2\sqrt{x}} = \int Q \times \mathrm{IF} \; \mathrm{dx} + C$$

$$\Rightarrow \quad y \, . e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \, . e^{2\sqrt{x}} \, \mathrm{dx} + C$$

$$\Rightarrow \quad y \, . e^{2\sqrt{x}} = \int \frac{\mathrm{dx}}{\sqrt{x}} + C$$

$$\Rightarrow \quad y \, . e^{2\sqrt{x}} = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$\Rightarrow \quad y \, . e^{2\sqrt{x}} = 2\sqrt{x} + C$$

Q.10. Solve the differential equation

$$(x^2 - 1)\frac{\mathrm{dy}}{\mathrm{dx}} + 2\mathrm{xy} = \frac{2}{x^2 - 1}, \text{ where } x \in (-\infty, -1) \bigcup (1, \infty)$$

Ans.

The given differential equation is $(x^2 - 1)\frac{\mathrm{dy}}{\mathrm{dx}} + 2\mathrm{xy} = \frac{2}{x^2-1}$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} + \frac{2x}{x^2 - 1} y = \frac{2}{(x^2 - 1)^2} \qquad \ldots(i)$$

This is a linear differential equation of the form

$\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{Py} = Q$, where $P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

$$\therefore \quad \mathrm{IF} = e^{\int P \, \mathrm{dx}} = e^{\int 2x/(x^2-1) \, \mathrm{dx}} = e^{\log \, (x^2-1)} = (x^2 - 1)$$

Multiplying both sides of (i) by $\mathrm{IF} = (x^2 - 1)$, we get $(x^2 - 1)\frac{\mathrm{dy}}{\mathrm{dx}} + 2\mathrm{xy} = \frac{2}{x^2-1}$

Integrating both sides, we get

$$y(x^2-1) = \int \frac{2}{x^2-1}\,dx + C \qquad [\text{Using } : y\ (\text{IF}) = \int Q.(\text{IF})dx + C]$$

$$\Rightarrow \quad y(x^2-1) = \frac{2}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

$$\Rightarrow \quad y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$$

This is the required solution.

Q.11. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$ given that y = 0 when x = 1.

Ans.

Given differential equation is $\frac{dy}{dx} = 1 + x + y + xy$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x) + y(1+x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x)(1+y)$$

$$\Rightarrow \quad \frac{dy}{1+y} = (1+x)dx$$

Integrating both sides, we get $\log|1+y| = \int(1+x)dx$

Integrating both sides, we get $\log|1+y| = \int(1+x)dx$

$$\Rightarrow \quad \log|1+y| = x + \frac{x^2}{2} + C, \text{ it is general solution.}$$

Putting $x = 1$, $y = 0$, we get

$$\log 1 = 1 + \frac{1}{2} + C$$

$$\Rightarrow \quad 0 = \frac{3}{2} + C$$

$$\Rightarrow \quad C = \frac{-3}{2}$$

Hence, particular solution is $\log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$.

Q.12. Solve the following differential equation:

$$x\log x\frac{dy}{dx}+y=\ 2\log x$$

Ans.

We have the differential equation

$$x\log x\frac{dy}{dx}+y=2\log x$$

$$\Rightarrow\quad \frac{dy}{dx}+\frac{1}{x\log x}.y=\frac{2}{x}$$

It is linear differential equation of the form

$\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{x\log x}$ and $Q=\frac{2}{x}$

Now, $IF=e^{\int pdx}=e^{\int\frac{1}{x\log x}dx}=e^{\log|\log x|}=\log x$

Hence, solution of given differential equation is $y\times IF=\int Q\times IF\,dx$

$$\Rightarrow\quad y\log x=\int\frac{2}{x}.\ \log x\ dx=2\int\frac{1}{x}.\log x\ dx=2\frac{(\log x)^2}{2}+C$$

$$\Rightarrow\quad y\log x=(\log x)^2+C$$

Q.13. Solve the differential equation $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x.$

Ans.

Given differential equation is $x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$

$$\Rightarrow\quad \frac{dy}{dx}+\left(\frac{1}{x.\log x}\right).y=\frac{2}{x^2}\qquad\text{(Divide each term by } x\log x)$$

It is in the form $\frac{dy}{dx}+Py=Q$ where $P=\frac{1}{x.\log x},Q=\frac{2}{x^2}$

$$\therefore\quad IF=e^{\int Pdx}=e^{\int\frac{dx}{x\log x}}$$

Put log $x = z \Rightarrow \frac{dx}{x} = dz = e^{\int \frac{1}{z} dz} = e^{\log z} = z = \log x$

$\therefore$ General solution is

$$y . \log x = \int \log x . \frac{2}{x^2} dx + C$$

$$\Rightarrow \quad y \log x = 2 \int \frac{\log x}{x^2} dx + C$$

Let $\log x = z \Rightarrow \frac{1}{x} dx = dz,$

Also $\quad \log x = z \Rightarrow x = e^z$

$$\therefore \quad y \log x = 2 \int \frac{z}{e^z} dz + C$$

$$\Rightarrow \quad y \log x = 2 \int z . e^{-z} dz + C$$

$$\Rightarrow \quad y \log x = 2 \left[z . \frac{e^{-z}}{-1} - \int \frac{e^{-z}}{-1} dz \right] + C$$

$$\Rightarrow \quad y \log x = 2[-ze^{-z} + \int e^{-z} dz] + C$$

$$\Rightarrow \quad y \log x = -2ze^{-z} - 2e^{-z} + C$$

$$\Rightarrow \quad y \log x = -2 \log x \, e^{-\log x} - 2e^{-\log x} + C$$

$$\Rightarrow \quad y \log x = -2 \log x . \frac{1}{x} - \frac{2}{x} + C \qquad \left[\because e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x} \right]$$

$$\Rightarrow \quad y \log x = -\frac{2}{x}(1 + \log x) + C$$

Q.14. Solve the following differential equation:

$$\text{cosec } x \log y \frac{dy}{dx} + x^2 y^2 = 0$$

Ans.

We have, $\text{cosec } x \log y \frac{dy}{dx} + x^2y^2 = 0$

$\Rightarrow \quad \text{cosec } x . \log y \frac{dy}{dx} = -x^2y^2$

$\Rightarrow \quad \frac{\log y . dy}{y^2} = -\frac{x^2 \, dx}{\text{cosec } x}$

$\Rightarrow \quad \int y^{-2} . \log y \, dy = -\int x^2 \sin x \, dx$

$\Rightarrow \quad \log y . \frac{y^{-2+1}}{-2+1} - \int \frac{1}{y} . \frac{y^{-2+1}}{-2+1} dy = -[x^2(-\cos x) - \int 2x(-\cos x) dx]$

$\Rightarrow \quad -\frac{1}{y} \log y + \int y^{-2} dy = x^2 \cos x - 2 \int x \cos x \, dx$

$\Rightarrow \quad -\frac{1}{y} \log y + \frac{y^{-2+1}}{-2+1} = x^2 \cos x - 2 [x \sin x - \int \sin x \, dx]$

$\Rightarrow \quad -\frac{1}{y} \log y - \frac{1}{y} = x^2 \cos x - 2x \sin + 2(-\cos x) + C$

$\Rightarrow \quad -\frac{1}{y}(\log y + 1) = x^2 \cos x - 2x \sin x - 2 \cos x + C$

Q.15. Solve the following differential equation:

$\frac{dy}{dx} + y = \cos x - \sin x$

Ans.

Given differential equation,

$\frac{dy}{dx} + y = \cos x - \sin x$ is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where

$P = 1, \; Q = \cos x - \sin x$

Here, $\text{IF} = e^{\int 1.\, dx} = e^x$

Its solution is given by

$$\Rightarrow \quad ye^x = \int e^x (\cos x - \sin x)\, dx$$

$$\Rightarrow \quad ye^x = \int e^x \cos x\, dx - \int e^x \sin x\, dx \qquad \text{(Integrating by parts)}$$

$$\Rightarrow \quad ye^x = e^x \cos x - \int -\sin x e^x\, dx - \int e^x \sin x\, dx$$

$$\therefore \quad ye^x = e^x \cos x + C$$

$$\Rightarrow \quad y = \cos x + Ce^{-x}$$

Q.16. Show that the differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0$ **is homogeneous. Find the particular solution of this differential equation, given that x = 1 when** $y = \frac{\pi}{2}$.

Ans.

Given differential equation is $x\frac{dy}{dx}\sin\frac{y}{x} + x - y\sin\frac{y}{x} = 0$

Dividing both sides by $x \sin \frac{y}{x}$, we get

$$\frac{dy}{dx} + \text{cosec}\,\frac{y}{x} - \frac{y}{x} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \text{cosec}\,\frac{y}{x} \qquad \ldots(i)$$

Let $F(x, y) = \frac{y}{x} - \text{cosec}\,\frac{y}{x}$

$$\therefore \quad F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \text{cosec}\,\frac{\lambda y}{\lambda x} = \lambda^0\left[\frac{y}{x} - \text{cosec}\,\frac{y}{x}\right] = \lambda^0 F(x, y)$$

Hence, differential equation (i) is homogeneous.

Let $y = \text{vx}$

$\Rightarrow \quad \frac{y}{x} = v$

$\Rightarrow \quad \frac{\text{dy}}{\text{dx}} = v + x \, . \frac{\text{dv}}{\text{dx}}$

Now, equation (i) becomes

$$v + x . \frac{\text{dv}}{\text{dx}} = \frac{\text{vx}}{x} - \text{cosec} \, \frac{\text{vx}}{x}$$

$$v + x . \frac{\text{dv}}{\text{dx}} = v - \text{cosec} \, v$$

$\Rightarrow \quad x . \frac{\text{dv}}{\text{dx}} = - \text{cosec} \, v$

$\Rightarrow \quad - \sin v \, \text{dv} = \frac{\text{dx}}{x}$

$\Rightarrow \quad - \int \sin v \, \text{dv} = \int \frac{\text{dx}}{x}$

$\Rightarrow \quad \cos v = \log |x| + C$

$\Rightarrow \quad \cos \frac{y}{x} = \log |x| + C \qquad \ldots (ii)$

Putting $y = \frac{\pi}{2}$, $x = 1$ in (ii), we get

$\therefore \quad \cos \frac{\pi}{2} = \log 1 + C$

$\Rightarrow \quad 0 = 0 + C \quad \Rightarrow \quad C = 0$

Hence, particular solution is

$$\cos \frac{\pi}{2} = \log |x| + 0 \qquad i.e., \qquad \cos \frac{y}{x} = \log |x|$$

Q.17. Solve the following differential equation:

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

Ans.

Given differential equation is,

$$\cos^2 x . \frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \quad \frac{dy}{dx} + y \sec^2 x = \tan x . \sec^2 x$$

Given differential equation is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$ where $P = \sec^2 x$, $Q = \tan x . \sec^2 x$.

$$IF = e^{\int P\, dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

$\therefore$ Solution is given by

$$e^{\tan x} y = \int \tan x . \sec^2 x . e^{\tan x}\, dx$$

Let $I = \int \tan x . \sec^2 x . e^{\tan x}\, dx$

Put $\tan x = t$, $\sec^2 x\, dx = dt$, we get

$$I = \int t\, e^t\, dt$$

$$\therefore \quad = t e^t - \int e^t dt = t e^t - e^t + C \qquad \text{[Integrating by parts]}$$

$$= \tan x\, e^{\tan x} - e^{\tan x} + C$$

Hence, $e^{\tan x} y = e^{\tan x}(\tan x - 1) + C$

$$\Rightarrow \quad y = \tan x - 1 + C e^{-\tan x}$$

Q.18. Find the particular solution of the differential equation satisfying the given condition $\frac{dy}{dx} = y \tan x,$ given that *y* = 1 when *x* = 0.

Ans.

We have $\frac{dy}{dx} = y \tan x \quad \Rightarrow \quad \frac{dy}{y} = \tan x \, dx$

By integrating both sides, we get

$$\int \frac{dy}{y} = \int \tan x \, dx$$

$$\log y = \log |\sec x| + C \qquad \ldots(i)$$

By putting $x = 0$ and $y = 1$ (as given) in (i), we get

$$\log 1 = \log (\sec 0) + C \quad \Rightarrow \quad C = 0$$

$\therefore \ (i) \Rightarrow \ \log y = \log |\sec x|$

$\Rightarrow$ Hence, the particular solution is $y = \sec x$

Q.19. Solve the differential equation:

$$\sqrt{1 + x^2 + y^2 + x^2y^2} + xy \frac{dy}{dx} = 0$$

Ans.

Given $\sqrt{1 + x^2 + y^2 + x^2y^2} + xy \frac{dy}{dx} = 0$

By simplifying the equation, we get

$$xy \frac{dy}{dx} = -\sqrt{1 + x^2 + y^2 + x^2y^2}$$

$$\Rightarrow \quad xy \frac{dy}{dx} = -\sqrt{(1 + x^2) + (1 + y^2)} = -\sqrt{(1 + x^2)}\sqrt{(1 + y^2)}$$

$$\Rightarrow \quad \frac{y}{\sqrt{(1+y^2)}}\mathrm{dy} = -\frac{\sqrt{(1+x^2)}}{x}\mathrm{dx}$$

Integrating both sides, we get

$$\int \frac{y}{\sqrt{(1+y^2)}}\mathrm{dy} = -\int \frac{\sqrt{(1+x^2)}}{x}\mathrm{dx} \qquad \ldots(i)$$

Let $1 + y^2 = t \quad \Rightarrow \quad 2y\,dy = dt,$ (For LHS)

and $1 + x^2 = m^2 \quad \Rightarrow \quad 2x\,dx = 2m\,dm \quad \Rightarrow \quad x\,dx = m\,dm$ (For RHS)

$$\therefore\ (i) \Rightarrow \quad \frac{1}{2}\int \frac{1}{\sqrt{t}}\mathrm{dt} = -\int \frac{m}{m^2-1}\,.m\,\mathrm{dm}$$

$$\Rightarrow \quad \frac{1}{2}\frac{t^{1/2}}{1/2} + \int \frac{m^2}{m^2-1}\mathrm{dm} = 0$$

$$\Rightarrow \quad \sqrt{t} + \int \frac{m^2+1-1}{m^2-1}\mathrm{dm} = 0$$

$$\Rightarrow \quad \sqrt{t} + \int \left(1 + \frac{1}{m^2-1}\right)\mathrm{dm} = 0$$

$$\Rightarrow \quad \sqrt{t} + m + \frac{1}{2}\log\left|\frac{m-1}{m+1}\right| = 0$$

Now, substituting these value of t and m, we get

$$\sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C = 0$$

Q.20. Show that the following differential equation is homogeneous and then solve it.

$$y\,\mathrm{dx} + x\log\left(\frac{y}{x}\right)\,\mathrm{dy} - 2x\,\mathrm{dy} = 0$$

Ans.

We have $y\,\mathrm{dx} + x\log\left(\frac{y}{x}\right)\,\mathrm{dy} - 2x\,\mathrm{dy} = 0$

Simplifying the above equation, we get

$$\left[x\log\left(\frac{y}{x}\right) - 2x\right]\mathrm{dy} = -y\,\mathrm{dx}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \qquad \ldots(i)$$

Let $F(x, y) = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$

$$F(\mu x, \mu y) = \frac{\mu y}{2\mu x + \mu x\log\left(\frac{\mu y}{\mu x}\right)} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} = \mu^{\circ} F(x, y)$$

$\therefore$ Function $F(x, y)$ is homogenous and hence the equation is homogeneous.

Let $y = vx \quad \Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dv}}{\mathrm{dx}}$

Substituting in equation (i), we get

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{vx}}{2x - x\log v}$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{v}{2 - \log v} - v$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{v\log v - v}{2 - \log v}$$

$$\Rightarrow \quad \frac{2 - \log v}{v\log v - v}\mathrm{dv} = \frac{\mathrm{dx}}{x}$$

Integrating both sides, we get

$$\int \frac{2-\log v}{v \log v - v} dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{1+(1-\log v)}{v(\log v - 1)} dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dv}{v(\log v - 1)} - \int \frac{dv}{v} = \int \frac{dx}{x}$$

Let $\log v - 1 = m \quad \Rightarrow \quad \frac{1}{v} dv = dm$

$$\Rightarrow \quad \int \frac{1}{m} dm - \int \frac{1}{v} dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \log|m| - \log|v| = \log|x| + \log|C|$$

$$\Rightarrow \quad \log\left|\frac{m}{v}\right| = \log|Cx| \quad \Rightarrow \quad \frac{m}{v} = Cx$$

$$\Rightarrow \quad (\log v - 1) = vCx$$

$$\Rightarrow \quad \left[\log\left(\frac{y}{x}\right) - 1\right] = Cy$$

which is the required solution.

Q.21. Solve the differential equation:

$$(x^2 + 1)\frac{dy}{dx} + 2xy = \sqrt{x^2 + 4}$$

Ans.

We have $(x^2+1)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$

Simplifying the above equation, we get

$$\frac{dy}{dx}+\frac{2x}{x^2+1}y=\frac{\sqrt{x^2+4}}{(x^2+1)}$$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$

where, $P=\frac{2x}{x^2+1}, Q=\frac{\sqrt{x^2+4}}{(x^2+1)}$

$$\therefore \quad IF=e^{\int P\,dx}=e^{\int\frac{2x}{x^2+1}dx}=e^{\log(x^2+1)}=(x^2+1)$$

Its solution is given by

$$(x^2+1)y=\int(x^2+1).\frac{\sqrt{x^2+4}}{(x^2+1)}dx=\int\sqrt{x^2+4}\,dx$$

$$\Rightarrow \quad (x^2+1)y=\frac{x}{2}\sqrt{x^2+4}+\frac{4}{2}\log\left|x+\sqrt{x^2+4}\right|+C$$

Q.22. Find the particular solution of the following differential equation satisfying the given condition:

$(3x^2+y)\frac{dx}{dy}=x, x>0$, **when $x=1, y=1$**

Ans.

We are given

$$(3x^2+y)\frac{dx}{dy}=x, x>0$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{x}{3x^2+y}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{3x^2+y}{x}=3x+\frac{y}{x}$$

$$\Rightarrow \quad \frac{dx}{dy}-\frac{1}{x}y=3x$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = -\frac{1}{x}, Q = 3x$

$\therefore \quad IF = e^{\int P\,dx} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

Its solution is given by

$\therefore \quad \frac{y}{x} = \int \frac{1}{x} 3x\,dx = 3\int dx$

$\Rightarrow \quad \frac{y}{x} = 3x + C$

$\Rightarrow \quad y = 3x^2 + Cx$

Putting $x = 1$, $y = 1$, we get

$\Rightarrow \quad 1 = 3 + C$

$\Rightarrow \quad C = -2$

$\therefore \quad y = 3x^2 - 2x$

Q.23. $(x^2 + y^2)\,dy = xy\,dx$**. If** $y(1) = 1$ **and** $y(x_0) = e$**, then find the value of** x_0**.**

Ans.

Given differential equation is $(x^2 + y^2)dy = xy\ dx$

It is also written as

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} \qquad \ldots(i)$$

Now, to solve let $y = vx$.

Differentiating $y = vx$ with respect to x, we get

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

Putting $y = \mathrm{vx}$ and $\frac{\mathrm{dy}}{\mathrm{dx}} = v + x\frac{\mathrm{dy}}{\mathrm{dx}}$ in (i), we get

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{x.\,\mathrm{vx}}{x^2+(\,\mathrm{vx}\,)^2}$$

$$v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{vx}^2}{x^2+v^2x^2}$$

$$\Rightarrow \quad v + x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{\mathrm{vx}^2}{x^2(1+v^2)}$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{v}{(1+v^2)} - v$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{v - v - v^3}{(1+v^2)}$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}} = \frac{-v^3}{(1+v^2)}$$

$$\Rightarrow \quad \frac{(1+v^2)\,\mathrm{dv}}{v^3} =- \frac{\mathrm{dx}}{x}$$

Integrating both sides, we get

$$\int\frac{(1+v^2)\,\mathrm{dv}}{v^3} =-\int\frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \int\frac{\mathrm{dv}}{v^3}+\int\frac{\mathrm{dv}}{v} =-\log|x| + C$$

$$\Rightarrow \quad -\frac{1}{2v^2} + \log|v| =-\log|x| + C$$

$$\Rightarrow \quad -\frac{x^2}{2y^2} + \log\left|\frac{y}{x}\right| =-\log|x| + C$$

$$\Rightarrow \quad -\frac{x^2}{2y^2} + \log|y|-\log|x| =-\log|x| + C$$

$$\Rightarrow \quad -\frac{x^2}{2y^2} + \log|y| = C \qquad ...(ii)$$

Given, $x = 1, y = 1$

$$\Rightarrow \quad -\frac{1}{2 \times 1} + \log|1| = C$$

$$\Rightarrow \quad -\frac{1}{2} = C \qquad [\because \log 1 = 0]$$

Now (ii) becomes

$$-\frac{x^2}{2y^2} + \log|y| = -\frac{1}{2}$$

$$\Rightarrow \quad \log|y| = \frac{x^2}{2y^2} - \frac{1}{2}$$

$$\Rightarrow \quad \log|y| = \frac{x^2 - y^2}{2y^2}$$

Putting $x = x_0$ and $y = e$ in (iii), we get

$$\log|e| = \frac{x_0^2 - e^2}{2e^2}$$

$$\Rightarrow \quad 1 = \frac{x_0^2 - e^2}{2e^2}$$

$$\Rightarrow \quad x_0^2 - e^2 = 2e^2$$

$$\Rightarrow \quad x_0^2 = 3e^2$$

$$\Rightarrow \quad x_0 = \sqrt{3}e$$

Q.24. Find the particular solution of the differential equation.

$\frac{dy}{dx} + y \tan x = 3x^2 + x^3 \tan x$, $x \neq \frac{\pi}{2}$, **given that $y = 0$ when $x = \frac{\pi}{3}$.**

Ans.

Given, $\frac{\mathrm{dy}}{\mathrm{dx}} + y\tan x = 3x^2 + x^3 \tan\ x$

$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} + \tan x.y = 3x^2 + x^3 \tan x$

Comparing the given differential equation with linear from

$$\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{Py} = Q, \text{ we get}$$

$$P = \tan x, Q = 3x^2 + x^3 \tan x.$$

$\therefore \quad \mathrm{IF} = e^{\int \tan x\ \mathrm{dx}} = e^{\log\ \sec\ x} = \sec\ x.$

Therefore, general solution is given by

$$y.\sec x = \int (3x^2 + x^3 \tan x).\sec x\ \mathrm{dx} + C$$

$$\Rightarrow \quad y.\sec x = \int 3x^2 \sec x\ \mathrm{dx} + \int x^3 \tan x.\ \sec x\ \mathrm{dx}\ +\ C$$

$$\Rightarrow \quad y\sec x = \int 3x^2 \sec x\ \mathrm{dx} + x^3.\sec x - \int 3x^2.\sec\ x\ \mathrm{dx} + C$$

$$\Rightarrow \quad y\sec x = x^3 \sec x + C$$

$$\Rightarrow \quad y = x^3 + C\cos x$$

Now $x = \frac{\pi}{3}, y = 0$

$$\therefore \quad 0 = \left(\frac{\pi}{3}\right)^3 + C.\cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad 0 = \frac{\pi^3}{27} + \frac{C}{2}$$

$$\Rightarrow \quad C = -\ \frac{2\pi^3}{27}$$

Hence required particular solution is

$$y = x^3 - \frac{2\pi^3}{27}\cos x.$$

Q.25. Show that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$ **is homogeneous and solve it.**

Ans.

Given, $(x - y)\frac{dy}{dx} = x + 2y$

By simplifying the above equation, we get

$$\frac{dy}{dx} = \frac{x+2y}{x-y} \qquad \ldots(i)$$

Let $F(x,y) = \frac{x+2y}{x-y}$

then $F(\lambda x,\lambda y) = \frac{\lambda x+2\lambda y}{\lambda x-\lambda y} = \frac{\lambda(x+2y)}{\lambda(x-y)} = \lambda \circ F(x,y)$

$F(x, y)$ is homogeneous function and hence given differential equation is homogeneous.

Now, let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting these values in equation (i), we get

$$v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+2v}{1-v} - v = \frac{1+2v - v+v^2}{1-v} = \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{1+v+v^2}\mathrm{dv} = \frac{\mathrm{dx}}{x}$$

By integrating both sides, we get

$$\int \frac{1-v}{1+v+v^2}\mathrm{dv} = \int \frac{\mathrm{dx}}{x} \qquad \ldots(ii)$$

LHS $\int \frac{1-v}{v^2+v+1}\mathrm{dv}$

Let $1 - v = A(2v+1) + B = 2Av + (A + B)$

Comparing coefficients of both sides, we get

$$2A = -1, \ A + B = 1$$

or $A = -\frac{1}{2}, \ B = \frac{3}{2}$

$$\therefore \quad \int \frac{1-v}{v^2+v+1}\mathrm{dv} = \int \frac{-\frac{1}{2}(2v+1)+\frac{3}{2}}{v^2+v+1}\mathrm{dv}$$

$$= -\frac{1}{2}\int \frac{2v+1}{v^2+v+1}\mathrm{dv} + \frac{3}{2}\int \frac{\mathrm{dv}}{v^2+v+1}$$

$$= -\frac{1}{2}\int \frac{2v+1}{v^2+v+1}\mathrm{dv} + \frac{3}{2}\int \frac{\mathrm{dv}}{\left(v+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= -\frac{1}{2}\log\left|v^2+v+1\right| + \frac{3}{2} \times \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

Now, substituting it in equation (ii), we get

$$-\frac{1}{2}\log|v^2+v+1|+\sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)=\log x+C$$

$$\Rightarrow \quad -\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right|+\sqrt{3}\tan^{-1}\left(\frac{\frac{2y}{x}+1}{\sqrt{3}}\right)=\log x+C$$

$$\Rightarrow \quad -\frac{1}{2}\log|x^2+\text{xy}+y^2|+\frac{1}{2}\log x^2+\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=\log x+C$$

$$\Rightarrow \quad -\frac{1}{2}\log|x^2+\text{xy}+y^2|+\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)=C$$

Q.26. Solve the following differential equation:

$$(x^3+x^2+x+1)\frac{\text{dy}}{\text{dx}}=2x^2+x$$

Ans.

We have $\quad (x^3+x^2+x+1)\frac{\text{dy}}{\text{dx}}=2x^2+x$

$$\Rightarrow \quad \frac{\text{dy}}{\text{dx}}=\frac{2x^2+x}{x^3+x^2+x+1}$$

$$\Rightarrow \quad \text{dy}=\frac{2x^2+x}{(x^2+1)(x+1)}\text{dx}$$

Integrating both sides, we get

$$\int \text{dy}=\int\frac{2x^2+x}{(x^2+1)(x+1)}\,\text{dx} \qquad ...(i)$$

$$\Rightarrow \quad \frac{2x^2+x}{(x^2+1)(x+1)}=\frac{A}{x+1}+\frac{\text{Bx}+C}{x^2+1}=A(x^2+1)(\text{Bx}+C)(x+1) \qquad \text{[By partial fraction]}$$

$$\Rightarrow \quad 2x^2 + x = x^2(A+B) + x(B+C) + (A+C)$$

Comparing coefficients of both the sides, we get

$A + B = 2,\ B + C = 1$ and $A + C = 0$

$$\Rightarrow \quad B = \frac{3}{2}, A = \frac{1}{2}, C = \frac{-1}{2}$$

$$\therefore \quad (i) \Rightarrow \quad y = \int \left[\frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1} \right] \mathrm{dx}$$

$$= \frac{1}{2} \int \frac{1}{x+1} \mathrm{dx} + \frac{3}{2} \int \frac{x}{x^2+1} \mathrm{dx} - \frac{1}{2} \int \frac{1}{x^2+1} \mathrm{dx}$$

$$y = \frac{1}{2} \log|x+1| + \frac{3}{4} \log|x^2+1| - \frac{1}{2} \tan^{-1} x + C$$

Q.27. Solve the following differential equation:

$$\mathbf{(1 + y^2)(1 + \log x)\ dx + x\ dy = 0}$$

Ans.

We have $(1 + y^2)(1 + \log x)\ dx + x\ dy = 0$

$$x\ dy = -(1 + y^2)(1 + \log x)\ dx$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{1+y^2} = -\frac{1+\log x}{x} \mathrm{dx}$$

Integrating both sides, we get

$$\int \frac{\mathrm{dy}}{1+y^2} = -\int \frac{1+\log x}{x} \mathrm{dx}$$

$$\Rightarrow \quad \tan^{-1} y = -\int z\, \mathrm{dz} \qquad \left[\text{Let } 1 + \log x = z \Rightarrow \frac{1}{x} \mathrm{dx} = \mathrm{dz}\right]$$

$$\Rightarrow \quad \tan^{-1} y = -\frac{z^2}{2} + C$$

$$\Rightarrow \quad \tan^{-1} y = -\frac{1}{2}(1 + \log x)^2 + C$$

Q.28. Solve the following differential equation:

$x\,dy - (y + 2x^2)\,dx = 0$

Ans.

We have $x\,dy - (y + 2x^2)\,dx = 0$

The given differential equation can be written as

$\Rightarrow \quad x\frac{dy}{dx} - y = 2x^2 \quad$ or $\quad \frac{dy}{dx} - \frac{1}{x}.y = 2x$

This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-1}{x}, Q = 2x$

$$IF = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$$

$\therefore$ Solution is $y.\frac{1}{x} = \int 2x.\frac{1}{x}dx$

$\Rightarrow \quad y.\frac{1}{x} = 2x + C \quad$ or $\quad y = 2x^2 + Cx$

Q.29. Solve the differential equation, $xdx + (y - x^3)\,dx = 0$.

Ans.

We have $x\,dy + (y - x^3)\,dx = 0$

$\Rightarrow \quad xdy =- (y - x^3)dx$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-y + x^3}{x}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x} + x^2$

$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{x}\right).y = x^2$

It is in the form of $\frac{dy}{dx} + Py = Q,$ where $P = \frac{1}{x}$ and $Q = x^2$

$\therefore \quad IF = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Hence, solution is $y.x = \int x.x^2 dx + C$

$xy = \frac{x^4}{4} + C \quad \Rightarrow \quad y = \frac{x^3}{4} + \frac{C}{x}$

Q.30. Find the particular solution of the following differential equation.

$x\frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$, **given that when $x = 2$, $y = \pi$**

Ans.

Given differential equation is $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} + \sin\left(\frac{y}{x}\right) = 0 \qquad ...(i)$

It is homogeneous differential equation.

Let $\frac{y}{x} = v \Rightarrow y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

Putting these values in (i), we get

$$v+x\frac{\mathrm{dv}}{\mathrm{dx}}-v+\sin v=0$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}}+\sin v=0$$

$$\Rightarrow \quad x\frac{\mathrm{dv}}{\mathrm{dx}}=-\sin v$$

$$\Rightarrow \quad \frac{\mathrm{dv}}{\sin v}=\frac{-\mathrm{dx}}{x}$$

$$\Rightarrow \quad \operatorname{cosec} v\,\mathrm{dv}=-\frac{\mathrm{dx}}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \int \operatorname{cosec} v\,\mathrm{dv}=-\int \frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \log|\operatorname{cosec} v-\cot v|=-\log|x|+C$$

$$\Rightarrow \quad \log\left|\operatorname{cosec}\frac{y}{x}-\cot\frac{y}{x}\right|+\log|x|=C \quad \ldots(i)$$

Putting $x = 2$, $y = \mathrm{p}$ we get

$$\Rightarrow \quad \log\left|\operatorname{cosec}\frac{\pi}{2}-\cot\frac{\pi}{2}\right|+\log 2=C$$

$$\Rightarrow \quad \log 1+\log 2=C \qquad [\because \log 1=0]$$

$$\Rightarrow \quad C=\log 2$$

Hence, particular solution, is

$$\log\left|\operatorname{cosec}\frac{y}{x}-\cot\frac{y}{x}\right|+\log|x|=\log 2$$

$$\Rightarrow \quad \log\left|x.\left(\operatorname{cosec}\frac{y}{x}-\cot\frac{y}{x}\right)\right|=\log 2$$

$$\Rightarrow \quad x\left(\operatorname{cosec}\frac{y}{x}-\cot\frac{y}{x}\right)=2$$

Q.31. Find the particular solution of the differential equation:

$(1-y^2)(1+\log x)\ dx + 2xy\ dy = 0$ given that $y = 0$ when $x = 1$.

Ans.

We have

$$(1-y^2)(1+\log x)\ dx + 2xy\ dy = 0,$$

$$\Rightarrow \quad 2xy\ dy = -(1-y^2)(1+\log x)\ dx$$

$$\Rightarrow \quad \frac{2y\ \mathrm{dy}}{1-y^2} = -\frac{(1+\log x)\,\mathrm{dx}}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \int \frac{2y\ \mathrm{dy}}{1-y^2} = -\int \frac{(1+\log x)}{x}\mathrm{dx}$$

$$\Rightarrow \quad -\log|1-y^2| = -\int \frac{(1+\log x)}{x}\mathrm{dx}$$

$$\Rightarrow \quad -\log|1-y^2| = -\int z\ \mathrm{dz} \qquad \left[\text{Let } 1+\log x = z \quad \Rightarrow \quad \tfrac{1}{x}\mathrm{dx} = \mathrm{dz}\right]$$

$$\Rightarrow \quad \log|1-y^2| = \frac{z^2}{2} + C$$

$$\Rightarrow \quad \log|1-y^2| = \frac{(1+\log x)^2}{2} + C$$

Putting $x = 1$ and $y = 0$, we get

$$\Rightarrow \quad \log 1 = \frac{(1+\log 1)^2}{2} + C$$

$$\Rightarrow \quad 0 = \frac{1}{2} + C \quad \Rightarrow \quad C = -\frac{1}{2}$$

Hence particular solution is

$$\log|1-y^2| = \frac{(1+\log x)^2}{2} - \frac{1}{2}$$

Q.32. Find the general solution of the following differential equation:

$$(1+y^2) + (x - e^{\tan^{1} y})\frac{dy}{dx} = 0$$

Ans.

We have $(1+y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$

$$\Rightarrow \quad (x - e^{\tan^{-1} y})\frac{dy}{dx} = -(1+y^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1+y^2}{x-e^{\tan^{-1} y}}$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{x-e^{\tan^{-1} y}}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{x}{1+y^2} + \frac{e^{\tan^{-1} y}}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{1+y^2}x = -\frac{e^{\tan^{-1} y}}{1+y^2}$$

It is in the form $\frac{dx}{dy} + Px = Q$.

Where $P = \frac{1}{1+y^2}$ and $Q = \frac{e^{\tan^{-1} y}}{1+y^2}$

$$\therefore \quad IF = e^{\int P.\ dy} = e^{\int \frac{1}{1+y^2}\ dy}$$
$$= e^{\tan^{-1} y}$$

Therefore, general solution is

$$x \,.\, e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \,.\, e^{\tan^{-1} y} dy + C$$

$$\Rightarrow \quad x \,.\, e^{\tan^{-1} y} = \int e^z \,.\, e^z dz + C \qquad \text{Let } \tan^{-1} y = z$$

$$\Rightarrow \quad x \,.\, e^{\tan^{-1} y} = \int e^{2z} dz + C \qquad \frac{1}{1+y^2} dy = dz$$

$$\Rightarrow \quad x \,.\, e^{\tan^{-1} y} = \frac{e^{2z}}{2} + C$$

$$\Rightarrow \quad x \,.\, e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + C$$

$$\Rightarrow \quad x = \frac{1}{2}\, e^{\tan^{-1} y} + C \,.\, e^{-\tan^{-1} y}$$

Q.33. Find the particular solution of differential equation

$: \frac{dy}{dx} = -\ \frac{x+y\ \cos\ x}{1+\sin\ x}$

given that *y* = 1 when *x* = 0.

Ans.

We have

$$\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{x+y\cos x}{1+\sin\ x}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{x}{1+\sin\ x} - \frac{y\cos x}{1+\sin\ x}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} + \frac{\cos x}{1+\sin\ x} y = -\frac{x}{1+\sin\ x}$$

Comparing it with linear form of differential equation $\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{Py} = Q$, we get

$$P = \frac{\cos x}{1+\sin x}, Q = -\frac{x}{1+\sin x}$$

Now $\mathrm{IF} = e^{\int \frac{\cos x}{1+\sin x}\mathrm{dx}} = e^{\log\ |1+\sin x\,|} = 1 + \sin x$

Therefore, general solution is

$$\begin{aligned} y(1+\sin x) &= \int -\frac{x}{1+\sin x}(1+\sin x)\mathrm{dx} + C \\ &= -\int x\,\mathrm{dx} + C \\ y(1+\sin x) &= -\frac{x^2}{2} + C \end{aligned}$$

Given $y = 1$ and $x = 0$

$$1(1 + \sin 0) = 0 + \mathrm{C}$$

$$\Rightarrow \quad C = 1$$

Hence, particular solution is

$$\Rightarrow \quad y(1+\sin x) = -\frac{x^2}{2} + 1$$

$$y = \frac{2-x^2}{2(1+\sin x)}$$

Q.34. Solve the following differential equation :

$$(\cot^{-1} y + x)\,dy = (1 + y^2)\,dx$$

Ans.

We have

$$(\cot^{-1} y + x)\,dy = (1 + y^2)\,dx$$

This can be written as

$$\frac{dx}{dy} = \frac{\cot^{-1} y + x}{1+y^2} = \frac{\cot^{-1} y}{1+y^2} + \frac{x}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{1}{1+y^2}\,.\,x = \frac{\cot^{-1} y}{1+y^2}$$

It is linear differential equation of the form

$$\frac{dx}{dy} + Px = Q, \text{ where } P = \frac{-1}{1+y^2} \text{ and } Q = \frac{\cot^{-1} y}{1+y^2}$$

$$\therefore \quad IF = e^{\int -\frac{1}{1+y^2} dy} = e^{\cot^{-1} y}$$

Therefore, required solution of differential equation is

$$x.e^{\cot^{-1} y} = \int \frac{\cot^{-1} y}{1+y^2}\,.\,e^{\cot^{-1} y} dy + C$$

$$\Rightarrow \quad x.e^{\cot^{-1} y} = I + C \qquad \ldots (i)$$

Where, $I = \int \frac{\cot^{-1} y}{1+y^2}\,.\,e^{\cot^{-1} y}\,dy$

Let $\cot^{-1} y = t$

$$-\frac{1}{1+y^2}\,\mathrm{dy} = \mathrm{dt}$$

$$\Rightarrow \quad \frac{1}{1+y^2}\,\mathrm{dy} = -\,\mathrm{dt}$$

$$\Rightarrow \quad I = -\int t\,.\,e^t\,\mathrm{dt} = -\,[t.e^t - \int e^t\,\mathrm{dt}\,] = -t\,.\,e^t + e^t$$

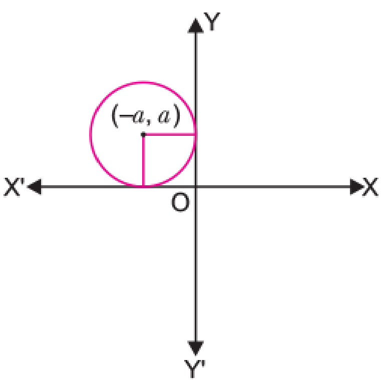
$$= e^t\,(1-t) = e^{\cot^{-1} y}\,(1-\cot^{-1} y)$$

Hence, required solution is

$$x\,.\,e^{\tan^{-1} y} = e^{\cot^{-1} y}\,(1-\cot^{-1} y) + C$$
$$x = (1-\cot^{-1} y) + \mathrm{C}e^{-\cot^{-1} y}$$

Q.35. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Ans. Let *C* denotes the family of circles in the second quadrant and touching the coordinate axes. Let (–*a*, *a*) be the coordinate of the centre of any member of this family (see figure).



Equation representing the family C is

$$(x + a)^2 + (y - a)^2 = a^2 \quad ...(i)$$

or $\quad x^2 + y^2 + 2ax - 2ay + a^2 = 0 \quad ...(ii)$

Differentiating equation (ii) with respect to x, we get

$$2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0$$

or $\quad x + y\frac{dy}{dx} = a\left(\frac{dy}{dx} - 1\right)$

or $\quad a = \frac{x+yy'}{y'-1} \qquad \left(y' = \frac{dy}{dx}\right)$

Substituting the value of a in equation (i), we get

$$\left[x + \frac{x+yy'}{y'-1}\right]^2 + \left[y - \frac{x+yy'}{y'-1}\right]^2 = \left[\frac{x+yy'}{y'-1}\right]^2$$

or $\quad [xy' - x + x + yy']^2 + [yy' - y - x - yy']^2 = [x + yy']^2$

or $\quad (x + y)^2 y'^2 + (x + y)^2 = (x + yy')^2$

or $\quad (x + y)^2 [(y')^2 + 1] = [x + yy']^2$, is the required differential equation representing the given family of circles.

Long Answer Questions-I (OIQ)

[4 Marks]

Q.1. Solve: $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Ans.

We have, $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\Rightarrow \quad \sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$

$\Rightarrow \quad \frac{\sec^2 x}{\tan x} \mathrm{dx} = -\frac{\sec^2 y}{\tan y} \mathrm{dy}$

$\Rightarrow \quad \int \frac{\sec^2 x}{\tan x} \mathrm{dx} = -\int \frac{\sec^2 y}{\tan y} \mathrm{dy}$

$\Rightarrow \quad \log|\tan x| = -\log|\tan y| + \log C$

$\Rightarrow \quad \log|(\tan x)(\tan y)| = \log C$

$\Rightarrow \quad |\tan x \tan y| = C$

Clearly, it is defined for $x \in \mathbb{R} - \{(2n+1)\pi/2 : n \in Z\}$

Hence, $|\tan x \tan y| = C$, where $x \in \mathbb{R} - \{(2n+1)\pi/2 : n \in Z\}$ is the solution of the given differential equation.

Q.2. Solve: $(x + 3y^2)\frac{\mathrm{dy}}{\mathrm{dx}} = y \; (y > 0)$

Ans.

Given differential equation is $(x + 3y^2)\frac{\mathrm{dy}}{\mathrm{dx}} = y, \; y0)$

We can write this as

$$\frac{\mathrm{dx}}{\mathrm{dy}} = \frac{x+3y^2}{y} = \frac{1}{y}.x + 3y$$

$$\Rightarrow \quad \frac{\mathrm{dx}}{\mathrm{dy}} + \left(-\frac{1}{y}\right).x = 3y$$

This is a linear equation of the form

$$\frac{\mathrm{dx}}{\mathrm{dy}} + \mathrm{Px} = Q, \text{ where } P = -\frac{1}{y}, Q = 3y$$

So, $\mathrm{IF} = e^{\int -\frac{1}{y}\mathrm{dy}} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$

Multiplying both sides by IF, we get

$$\frac{1}{y} \times \frac{dx}{dy} - \frac{1}{y^2} x = 3$$

$$\Rightarrow \quad \frac{d}{dy}\left(\frac{1}{y} . x\right) = 3$$

Integrating both sides, with respect to y, we get

$$\frac{1}{y} . x = 3y + C$$

Hence, $x = 3y^2 + Cy$ is the required solution.

Q.3. Solve $\frac{dy}{dx} + y \sec x = \tan x.$

Ans.

The given differential equation is

$$\frac{dy}{dx} + (\sec x) y = \tan x \qquad ...(i)$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \sec x$ and $Q = \tan x$

$$\therefore \quad IF = e^{\int P\, dx} = e^{\int \sec x\, dx} = e^{\log (\sec x + \tan x)} = (\sec x + \tan x)$$

Multiplying both sides of (i) by IF = $(\sec x + \tan x)$, we get

$$(\sec x + \tan x)\frac{dy}{dx} + y \sec x(\sec x + \tan x) = \tan x(\sec x + \tan x)$$

Integrating both sides, we get

$$y(\sec x+\tan x)=\int \tan x(\sec x+\tan x)dx+C \quad [\text{Using} : y(\text{IF})=\int Q.(\text{IF})dx+C]$$

$$\Rightarrow \quad y(\sec x+\tan x)=\int (\tan x\sec x+\tan^2 x)dx+C$$

$$\Rightarrow \quad y(\sec x+\tan x)=\int (\tan x\sec x+\sec^2 x-1)dx+C$$

$\Rightarrow \quad y(\sec x+\tan x)=\sec x+\tan x-x+C$, which is the required solution.

Q.4. Solve : $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$

Ans.

We have,

$$\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$$

$$\Rightarrow \quad dy=\left(e^{x-y}+x^2e^{-y}\right)dx$$

$$\Rightarrow \quad e^y dy=\left(e^x+x^2\right)dx$$

Integrating both sides, we get

$$\Rightarrow \quad \int e^y dy=\int (e^x+x^2)dx$$

$\Rightarrow \quad e^y=e^x+\frac{x^3}{3}+C$, which is the required solution.

Q.5. Form the differential equation representing the family of curves $y^2 - 2ay + x^2 = a^2$, where a is an arbitrary constant.

Ans.

Given family of curves $y^2 - 2ay + x^2 = a^2$ ...(i)

Differentiating with respect to x, we get

$$2y\frac{dy}{dx} - 2a\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow \quad (y-a)\frac{dy}{dx} + x = 0$$

$$(y-a)\frac{dy}{dx} = -x$$

$$\Rightarrow \quad y - a = -x.\frac{dx}{dy}$$

$$\Rightarrow \quad a = \left(y + x\frac{dx}{dy}\right)$$

Substituting the value of a, in (i), we get

$$y^2 - 2\left(y + x\frac{dx}{dy}\right)y + x^2 = \left(y + x\frac{dx}{dy}\right)^2$$

$$\Rightarrow \quad y^2 - 2y^2 - 2xy\frac{dx}{dy} + x^2 = y^2 + x^2\left(\frac{dx}{dy}\right)^2 + 2xy\frac{dx}{dy}$$

$$\Rightarrow \quad (x^2 - y^2) - 2xy\frac{dx}{dy} = y^2 + x^2\left(\frac{dx}{dy}\right)^2 + 2xy\frac{dx}{dy}$$

$$\Rightarrow \quad (x^2 - 2y^2) - 4xy\frac{dx}{dy} = x^2\left(\frac{dx}{dy}\right)^2 \qquad ...(ii)$$

Let $\frac{dy}{dx} = p \Rightarrow \frac{dx}{dy} = \frac{1}{p}$

Therefore, (ii) becomes, $(x^2 - 2y^2) - 4xy\frac{1}{p} = x^2\left(\frac{1}{p}\right)^2$

$$\Rightarrow \quad p^2(x^2 - 2y^2) - 4xyp = x^2$$

$$\Rightarrow \quad p^2(x^2 - 2y^2) - 4xyp - x^2 = 0, \text{ where } p = \frac{dy}{dx}.'$$

Q.6. Find the general solution of the following differential equation:

$$x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

Ans.

Given differential equation is $x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$

$$\frac{dy}{dx} = \frac{y \cos y/x + x}{x \cos y/x} \qquad \ldots(i)$$

It is homogeneous differential equation.

Let $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$(i) \quad \Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx \cos v + x}{x \,.\, \cos v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \cos v + 1}{\cos v} - v$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v \cos v + 1 - v \cos v}{\cos v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{\cos v}$

$\Rightarrow \quad \cos v\,dv = \frac{dx}{x}$

Integrating both sides

$\Rightarrow \quad \sin v = \log|x| + C$

$\Rightarrow \quad \sin\frac{y}{x} = \log|x| + C$

Long Answer Questions-II

[6 Marks]

Q.1. Solve the following differential equation: $3\mathbf{e}^{\mathbf{x}}\, tan\, \mathbf{y}\, \mathbf{dx} + (2 - \mathbf{e}^{\mathbf{x}})\sec^2 \mathbf{y}\, \mathbf{dy} = 0$**, given that when** $x = 0, y = \frac{\pi}{4}$

Ans.

Given, $3e^x \tan y\, dx + (2 - e^x) \sec^2 y\, dy = 0$

$\Rightarrow \quad (2 - e^x) \sec^2 y\, dy = - 3e^x \tan y\, dx$

$\Rightarrow \quad \frac{\sec^2 y}{\tan y} dy = \frac{-3e^x}{2 - e^x} dx$

$\Rightarrow \quad \int \frac{\sec^2 y\, dy}{\tan y} = 3 \int \frac{-e^x\, dx}{2 - e^x}$

$\Rightarrow \quad \log |\tan y| = 3 \log |2 - e^x| + \log C$

$\Rightarrow \quad \log |\tan y| = \log |C.\, (2 - e^x)^3|$

$\Rightarrow \quad \tan y = C\, (2 - e^x)^3$

Putting $x = 0, y = \frac{\pi}{4}$, we get

$\Rightarrow \quad \tan\frac{\pi}{4} = C(2 - e^0)^3$

$\Rightarrow \quad 1 = C(2 - 1)^3$

$\Rightarrow \quad 1 = C$

Therefore, particular solution is

tan $y = (2 - e^x)^3$.

Q.2. Solve: $x\ dy - y\, dx = \sqrt{x^2 + y^2}\, dx$

Ans.

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x}, x \neq 0$$

Clearly, it is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in it, we get

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2+v^2x^2}+vx}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{1}{\sqrt{1+v^2}} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \quad \log|v + \sqrt{1+v^2}| = \log|x| + \log C$$

$$\Rightarrow \quad |v + \sqrt{1+v^2}| = |Cx|$$

$$\Rightarrow \quad \left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = |Cx| \qquad [\because v = y/x]$$

$$\Rightarrow \quad \{y + \sqrt{x^2+y^2}\}^2 = C^2x^4 \qquad \text{[Squaring both sides]}$$

Hence, $\{y + \sqrt{x^2+y^2}\}^2 = C^2x^4$ gives the required solution.

Q.3. Find the particular solution of the differential equation $(1+x^3)\frac{dy}{dx} + 6x^2y = (1+x^2)$, **given that *y* = 1 when *x* = 1.**

Ans.

The given differential equation is

$$(1+x^3)\frac{dy}{dx}+6x^2y=(1+x^2)$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{6x^2}{(1+x^3)}y=\frac{1+x^2}{1+x^3}$$

It is in the form of $\frac{dy}{dx}+Py=Q$, where $P=\frac{6x^2}{(1+x^3)}, Q=\frac{1+x^2}{1+x^3}$

$$\therefore \quad IF=e^{\int P\ dx}=e^{\int \frac{6x^2}{1+x^3}\ dx}$$

$$=e^{2\int \frac{3x^2}{1+x^3}\ dx}=e^{2\int \frac{dt}{t}} \qquad [\text{Let } 1+x^3=t\Rightarrow 3x^2dx=dt]$$

$$=e^{2\log t}=e^{\log t^2}=t^2$$

$$=(1+x^3)^2$$

Therefore, general solution is

$$y.(1+x^3)^2=\int \frac{1+x^2}{1+x^3}\times(1+x^3)^2dx+C$$

$$=\int(1+x^2)(1+x^3)dx+C=\int(x^5+x^3+x^2+1)dx+C$$

$$y.(1+x^3)^2=\frac{x^6}{6}+\frac{x^4}{4}+\frac{x^3}{3}+x+C$$

Putting $y=1$, $x=1$, we get

$$\therefore \quad 4=\frac{1}{6}+\frac{1}{4}+\frac{1}{3}+1+C$$

$$\Rightarrow \quad C=4-\frac{1}{6}-\frac{1}{4}-\frac{1}{3}-1=\frac{9}{4}$$

Required particular solution is $y(1+x^3)^2=\frac{x^6}{6}+\frac{x^4}{4}+\frac{x^3}{3}+x+\frac{9}{4}$.

Q.4. Show that the differential equation $(xe^{\frac{y}{x}}+y)\,dx=x\,dy$ is homogeneous. Find the particular solution of this differential equation, given that *x* = 1 when *y* = 1.

Ans.

Given differential equation is,

$$\left(x.e^{\frac{y}{x}}+y\right)\mathrm{dx}=\mathrm{xdy}$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}}=\frac{x.e^{\frac{y}{x}}+y}{x} \qquad \ldots(i)$$

Let $F(x,y)=\frac{x.e^{\frac{y}{x}}+y}{x}$

$$\therefore \quad F(\lambda x,\lambda y)=\frac{\lambda x.e^{\frac{\lambda y}{\lambda x}}+\lambda y}{\lambda x}=\lambda^0\frac{x.e^{\frac{y}{x}}+y}{x}=\lambda^0F(x,y)$$

Hence, given differential equation (i) is homogenous.

Let $y=vx \quad \Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}}=v+x.\frac{\mathrm{dv}}{\mathrm{dx}}$

Now, given differential equation (i) would become

$$v+x\frac{\mathrm{dv}}{\mathrm{dx}}=\frac{x.e^{\frac{\mathrm{vx}}{x}}+\mathrm{vx}}{x}$$

$$\Rightarrow \quad v+x.\frac{\mathrm{dv}}{\mathrm{dx}}=e^v+v$$

$$\Rightarrow \quad x.\frac{\mathrm{dv}}{\mathrm{dx}}=e^v$$

$$\frac{\mathrm{dv}}{e^v}=\frac{\mathrm{dx}}{x}$$

$$\Rightarrow \quad \int e^{-v}\mathrm{dv}=\int\frac{\mathrm{dv}}{v}$$

$$\Rightarrow \quad \frac{e^{-v}}{-1}=\log x+C$$

$$-e^{-\frac{y}{x}}=\log x+C$$

$$\Rightarrow \quad -\frac{1}{e^{\frac{y}{x}}} = \log x + C$$

$$\Rightarrow \quad e^{\frac{y}{x}} . \log x + \mathrm{C}e^{\frac{y}{x}} + 1 = 0$$

Putting $x = 1$, $y = 1$, we get

$$\therefore \quad e \log 1 + Ce + 1 = 0$$

$$\Rightarrow \quad C = -\frac{1}{e}$$

$\therefore$ The required particular solution is

$$e^{\frac{y}{x}} . \log x - \frac{1}{e} e^{\frac{y}{x}} + 1 = 0 \quad \text{or} \quad e^{\frac{y}{x}} \log x - e^{\frac{y}{x} - 1} + 1 = 0$$

Q.5. Show that the differential equation $\left[x \sin^2\left(\frac{y}{x}\right) - y\right]\mathrm{dx} + x\,\mathrm{dy} = 0$ is homogeneous. Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when *x* = 1.

Ans.

Given differential equation is,

$$\left[x \sin^2\left(\frac{y}{x}\right) - y\right] \mathrm{dx} + x\,\mathrm{dy} = 0$$

$$\Rightarrow \quad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x} \qquad \ldots(i)$$

Let $F(x,y) = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x}$

Then $F(\lambda x, \lambda y) = \frac{\lambda y - \lambda x \sin^2 \frac{\lambda y}{\lambda x}}{\lambda x} = \lambda^0 \frac{y - x \sin^2 \frac{y}{x}}{x} = \lambda^0 F(x,y)$

Hence, differential equation (i) is homogeneous.

Now, let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Putting these value in (i), we get

$$\frac{dv}{dx} = \frac{vx - x\sin^2\frac{vx}{x}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{x\{v - \sin^2 v\}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad \frac{dv}{\sin^2 v} = -\frac{dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \quad \int \operatorname{cosec}^2 v\,dv = -\int \frac{1}{x}dx$$

$$\Rightarrow \quad -\cot v = -\log x + C$$

$$\Rightarrow \quad \log x - \cot\left(\frac{y}{x}\right) = C \qquad \ldots(ii)$$

Putting $y = \frac{\pi}{4}$ and $x = 1$ in (ii), we get

$$\log 1 - \cot\frac{\pi}{4} = C$$

$$\Rightarrow \quad 0 - 1 = C$$

$$\Rightarrow \quad C = -1$$

Hence, particular solution is

$$\log x - \cot\left(\frac{y}{x}\right) = -1$$

$$\Rightarrow \quad \log x - \cot\left(\frac{y}{x}\right) + 1 = 0$$

Q.6. Find the differential equation of the family of curves $(x-h)^2+(y-k)^2=r^2$, where *h* and *k* are arbitrary constants.

Ans.

Given family of curve is:

$$(x-h)^2+(y-k)^2=r^2 \qquad \ldots(i)$$

Differentiating with respect to x, we get

$$\Rightarrow \quad 2(x-h)+2(y-k).\frac{dy}{dx}=x$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{x-h}{y-k} \qquad \ldots(ii)$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2}=-\left\{\frac{(y-k)-(x-h).\frac{dy}{dx}}{(y-k)^2}\right\}=-\left\{\frac{(y-k)+(x-h).\frac{x-h}{y-k}}{(y-k)^2}\right\} \qquad [\text{From } (ii)]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-\left\{\frac{(y-k)^2+(x-h)^2}{(y-k)^3}\right\}=-\frac{r^2}{(y-k)^3} \qquad \ldots(iii) \qquad [\text{From } (i)]$$

From (ii) $\left(\frac{dy}{dx}\right)^2=\left(\frac{x-h}{y-k}\right)^2$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2=\frac{(x-h)^2}{(y-k)^2}$$

Adding 1 both the sides, we get

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2+1=\frac{(x-h)^2}{(y-k)^2}+1=\frac{(x-h)^2+(y-k)^2}{(y-k)^2}$$

Putting exponent (power) $\frac{3}{2}$ both sides, we get

$$\Rightarrow \quad \left[\left(\frac{dy}{dx}\right)^2+1\right]^{\frac{3}{2}}=\left[\frac{r^2}{(y-k)^2}\right]^{\frac{3}{2}}=\frac{r^3}{(y-k)^3}$$

$$\Rightarrow \quad \left[\left(\frac{dy}{dx}\right)^2+1\right]^{\frac{3}{2}}=r.\frac{r^2}{(y-k)^3}=-r\frac{d^2y}{dx^2} \qquad [\text{Using } (iii)]$$

$$\Rightarrow \quad r\frac{d^2y}{dx^2}+\left[\left(\frac{dy}{dx}\right)^2+1\right]^{3/2}=0$$

Q.7. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$ **given that** $y = \frac{\pi}{2}$ **when** $x = 1$.

Ans.

Given differential equation is $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y+y\cos y}$

$\Rightarrow \quad (\sin y + y\cos y)\, dy = x\,(2\log x + 1)\,dx$

$\Rightarrow \quad \int \sin y\, dy + \int y\cos y\, dy = 2\int x\log x\, dx + \int x\, dx$

$\Rightarrow \quad \int \sin y\, dy + [y\sin y - \int \sin y\, dy] = 2\left[\log x \frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}\, dx\right] + \int x\, dx$

$\Rightarrow \quad \int \sin y\, dy + y\sin y - \int \sin y\, dy = x^2\log x - \int x\, dx + \int x\, dx + C$

$\Rightarrow \quad y\sin y = x^2\log x + C \qquad \ldots (i)$

It is general solution.

For particular solution, we put $y = \frac{\pi}{2}$ when $x = 1$

(i) becomes $\frac{\pi}{2}\sin\frac{\pi}{2} = 1\,.\,\log 1 + C$

$\frac{\pi}{2} = C \qquad [\because \log 1 = 0]$

Putting the value of C in (i), we get the required particular solution

$y\sin y = x^2\log x + \frac{\pi}{2}$

Q.8. Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2+y^2}{2xy}$, **is given by** $x^2 - y^2 = Cx$.

Ans.

We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$

Therefore, $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}} \qquad ...(i)$$

Clearly, equation (i) is a homogeneous differential equation. To solve it we make substitution

$$y = vx$$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting the value of y and $\frac{dy}{dx}$ in equation (i), we get

$$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

or $\quad x\frac{dv}{dx} = \frac{1-v^2}{2v}$

$$\Rightarrow \quad \frac{2v}{1-v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \quad \frac{2v}{v^2-1}dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{2v}{v^2-1}\, dv = -\int \frac{1}{x}\, dx$$

or $\quad \log|v^2 - 1| = -\log|x| + \log|C_1|$

or $\quad \log|(v^2 - 1)(x)| = \log|C_1|$

or $\quad (v^2 - 1)\, x = \pm C_1$

Replacing v by $\frac{y}{x}$ we get

$$\left(\frac{y^2}{x^2} - 1\right)x = \pm C_1$$

$$\Rightarrow \quad (y^2 - x^2) = \pm C_1 x$$

$$\Rightarrow \quad x^2 - y^2 = Cx \qquad (\text{where } \pm C_1 = C)$$