Very Short Answer Questions (PYQ)

[1 Mark]

Q1. If a line has direction ratios 2, –1, –2, then what are its direction cosines?

Ans.

Here direction ratios of line are 2, −1, −2

$\therefore$ Direction cosines of line are $\frac{2}{\sqrt{2^2+(-1)^2+(-2)^2}}, \frac{-1}{\sqrt{2^2+(-1)^2+(-2)^2}}, \frac{-2}{\sqrt{2^2+(-1)^2+(-2)^2}}$

i.e., $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$

Note: If *a, b, c* are the direction ratios of a line, the direction cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Q.2. If the equation of a line AB is $\frac{x-3}{1} = \frac{y+2}{-2} = \frac{z-5}{4}$ then find the direction ratios of line parallel to *AB*.

Ans. The direction ratios of line parallel to *AB* is 1, –2 and 4.

Q.3. Write the direction cosine of a line equally inclined to the three coordinate axes.

Ans.

Any line equally inclined to coordinate axes will have direction cosines *l, l, l*

$\therefore \quad l^2 + l^2 + l^2 = 1$

$3l^2 = 1 \quad \Rightarrow \quad l = \pm\frac{1}{\sqrt{3}}$

$\therefore$ Direction cosines are $+\frac{1}{\sqrt{3}}, +\frac{1}{\sqrt{3}}, +\frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

Q.4. Write the distance of the following plane from the origin.

$2x - y + 2z + 1 = 0$

Ans.

We have given plane

$2x - y + 2z + 1 = 0$

Distance from origin = $\left|\frac{(2\times0) - (1\times0)+(2\times0)+1}{\sqrt{(2)^2+(-1)^2+(2)^2}}\right| = \left|\frac{1}{\sqrt{4+1+4}}\right| = \frac{1}{3}$

Q.5. Write the cartesian equation of the following line given in vector form.

$\vec{r} = (2\hat{i} + \hat{j} - 4\hat{k}) + \lambda(\hat{i} - \hat{j} - \hat{k})$

Ans.

Vector form of a line is given as

$\vec{r} = 2\hat{i} + \hat{j} - 4\hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$

Direction ratios of above equation are $(1, -1, -1)$ and point through which the line passes is $(2, 1, -4)$.

$\therefore$ Cartesian equation is

$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

i.e., $\frac{x - 2}{1} = \frac{y - 1}{-1} = \frac{z+4}{-1}$ or $x - 2 = 1 - y = -z - 4$

Q.6. Write the direction cosines of a line parallel to *z*-axis.

Ans.

The angle made by a line parallel to *z*-axis with *x, y* and *z*-axis are 90°, 90° and 0° respectively.

∴ The direction cosines of the line are cos 90º, cos 90º, cos 0º *i.e.*, 0, 0, 1.

Q.7. Write the cartesian equation of a plane, bisecting the line segment joining the points A(2, 3, 5) and B(4, 5, 7) at right angles.

Ans.

One point of required plane = mid point of given line segment.

$$= \left(\frac{2+4}{2}, \frac{3+5}{2}, \frac{5+7}{2}\right) = (3, 4, 6)$$

Also Dr's of normal to the plane = $(4-2), (5-3), (7-5)$

$= 2, 2, 2$

Therefore, required equation of plane is

$$2(x-3) + 2(y-4) + 2(z-6) = 0$$

$$2x + 2y + 2z = 26 \text{ or } x + y + z = 13$$

Q.8. Write the vector equation of the plane, passing through the point (*a*, *b*, *c*) and parallel to the plane $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 2.$

Ans.

Since, the required plane is parallel to plane $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 2.$

$\therefore$ Normal of required plane is normal of given plane.

$\Rightarrow$ Normal of required plane $= \hat{i} + \hat{j} + \hat{k}$

$\therefore$ Required vector equation of plane

$$\{\vec{r} - (a\hat{i} + b\hat{j} + c\hat{k})\}.(\hat{i} + \hat{j} + \hat{k}) = 0$$

Q.9. Find the angle between the lines 2*x* = 3*y* = – *z* and 6*x* = – *y* = – 4*z*.

Ans.

Given lines are

2x = 3y = – z and 6x = – y = – 4z which may be written as

$$\frac{x-0}{\frac{1}{2}} = \frac{y-0}{\frac{1}{3}} = \frac{z-0}{-1} \quad \dots (i) \quad \text{and} \quad \frac{x-0}{\frac{1}{6}} = \frac{y-0}{-1} = \frac{z-0}{\frac{-1}{4}}$$

Parallel vectors $\vec{b_1}$ and $\vec{b_2}$ of (i) and (ii) respectively are

$$\vec{b_1} = \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} - \hat{k} \text{ and } \vec{b_2} = \frac{1}{6}\hat{i} - \hat{j} - \frac{1}{4}\hat{k}$$

$\therefore$ Angle between lines (i) and (ii) = Angle between $\vec{b_1}$ and $\vec{b_2}$

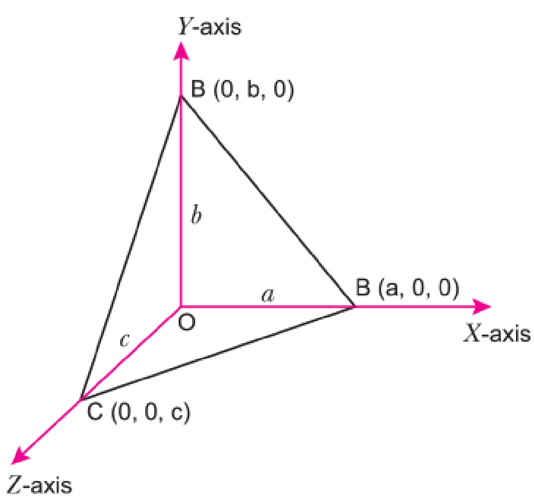
$$= \cos^{-1}\left(\left|\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|\right) = \cos^{-1}\left(\left|\frac{\left(\frac{1}{2}\hat{i}+\frac{1}{3}\hat{j}-\hat{k}\right).\left(\frac{1}{6}\hat{i}-\hat{j}-\frac{1}{4}\hat{k}\right)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{3}\right)^2+(-1)^2}\sqrt{\left(\frac{1}{6}\right)^2+(-1)^2+\left(-\frac{1}{4}\right)^2}}\right|\right)$$

$$= \cos^{-1}\left(\left|\frac{\frac{1}{12}-\frac{1}{3}+\frac{1}{4}}{\sqrt{\frac{1}{4}+\frac{1}{9}+1}\sqrt{\frac{1}{36}+1+\frac{1}{16}}}\right|\right) = \cos^{-1}0 = \frac{\pi}{2}$$

Q.10. Find the sum of the intercepts cut off by the plane 2*x* + *y* – *z* = 5, on the coordinate axes.

Ans.

Let a, b, c be the intercepts cut off by the plane



$2x + y - z = 5 \qquad \ldots (i)$ on x, y and z-axis respectively.

$\Rightarrow A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ satisfy the equation (i)

Hence, $2a + 0 - 0 = 5 \Rightarrow a = \frac{5}{2}$

$2 \times 0 + b - 0 = 5 \Rightarrow b = 5$

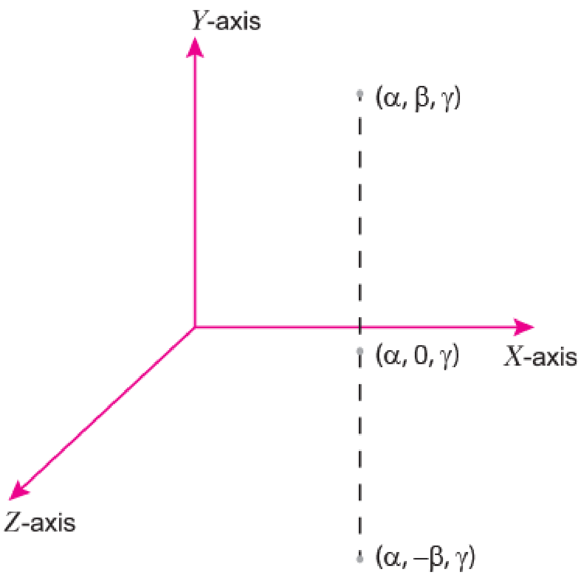
$2 \times 0 + 0 - c = 5 \Rightarrow c = -5$

$\therefore \quad a + b + c = \frac{5}{2} + 5 - 5 = \frac{5}{2}$

Q.11. Write the coordinates of the point which is the reflection of the point in the XZ-plane.

Ans.

The reflection of the point (α,β,γ) in the *XZ* plane is (α,–β,γ).



Q.12. Find the distance between the planes

$\vec{r}.(2\hat{i} - 3\hat{j} + 6\hat{k}) - 4 = 0$ and $\vec{r}.(6\hat{i} - 9\hat{j} + 18\hat{k}) + 30 = 0$

Ans.

Given two planes are

$\vec{r}.(2\hat{i} - 3\hat{j} + 6\hat{k}) - 4 = 0$ and $\vec{r}.(6\hat{i} - 9\hat{j} + 18\hat{k}) + 30 = 0$

Given planes may be written in cartesian form as

$2x - 3y + 6z - 4 = 0 \quad \ldots (i)$

$6x - 9y + 18z + 30 = 0 \quad \ldots (ii)$

Let $P(x_1, y_1, z_1)$ be a point on plane (i)

$\therefore \; 2x_1 - 3y_1 + 6z_1 - 4 = 0$

$\Rightarrow \; 2x_1 - 3y_1 + 6z_1 = 4 \quad \ldots (iii)$

The length of the perpendicular from $P(x_1, y_1, z_1)$ to plane (ii)

$= \left| \frac{6x_1 - 9y_1 + 18z_1 + 30}{\sqrt{6^2 + (-9)^2 + 18^2}} \right| = \left| \frac{3(2x_1 - 3y_1 + 6z_1) + 30}{\sqrt{36 + 81 + 324}} \right|$

$= \left| \frac{3 \times 4 + 30}{\sqrt{441}} \right| = \left| \frac{42}{21} \right| = 2$ [Using (iii)]

Q.13. Write the equation of a plane which is at a distance of $5\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axes.

Ans.

Obviously, a vector equally inclined to co-ordinate axes is given by $\hat{i}+\hat{j}+\hat{k}$

$\therefore$ Unit vector equally inclined to co-ordinate axes $=\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$

Therefore, required equation of plane is

$$\vec{r}\cdot\left\{\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\right\}=5\sqrt{3}$$

$$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=15 \quad \text{or} \quad x+y+z=15$$

Q.14. If a line makes angles 90° and 60° respectively with the positive directions of *x* and *y* axes, find the angle which it makes with the positive direction of *z*-axis.

Ans.

Let the angle made by line with positive direction of *z*-axis be then,

we know that

$$\cos^2 90+\cos^2 60+\cos^2\theta=1$$

$$\Rightarrow \quad 0+\left(\frac{1}{2}\right)^2+\cos^2\theta=1 \quad \Rightarrow \quad \frac{1}{4}+\cos^2\theta=1$$

$$\Rightarrow \quad \cos^2\theta=1-\frac{1}{4} \quad \Rightarrow \quad \cos^2\theta=\frac{3}{4}$$

$$\Rightarrow \quad \cos\theta=\pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \theta=60^\circ \text{ or } \frac{\pi}{3} \text{ if } \cos\theta=\frac{\sqrt{3}}{2} \text{ and } \theta=150^\circ \text{ or } \frac{5\pi}{6} \text{ if } \cos\theta=-\frac{\sqrt{3}}{2}$$

Q.15. Find the distance between the planes 2*x* – *y* + 2*z* = 5 and 5*x* – 2.5*y* + 5*z* = 20.

Ans.

Let $P(x_1, y_1, z_1)$ be any point on plane

$$2x - y + 2z = 5 \ \ldots (i)$$

$$\Rightarrow \ 2x_1 - y_1 + 2z_1 = 5$$

Now distance of point $P(x_1, y_1, z_1)$ from plane $5x - 2.5y + 5z = 20$ is given by

$$d = \left|\frac{5x_1 - 2.5y_1 + 5z_1 - 20}{\sqrt{5^2 + (2.5)^2 + (5)^2}}\right| = \left|\frac{2.5(2x_1 - y_1 + 2z_1 - 8)}{\sqrt{25 + 6.25 + 25}}\right|$$
$$= \left|\frac{2.5(5 - 8)}{\sqrt{56.25}}\right|$$
$$= \frac{7.5}{7.5} = 1 \text{ unit}$$

Very Short Answer Questions (OIQ)

[1 Mark]

Q.1. Write the angle between lines

$$\frac{x - 2}{3} = \frac{y + 1}{-2}, z = 2 \text{ and } \frac{x - 1}{1} = \frac{2y + 3}{3} = \frac{z + 5}{2}.$$

Ans.

From two given lines, we have direction ratios 3, –2, 0 and 1, $\frac{3}{2}$, 2 respectively.

Therefore, cos q = $\frac{3 \times 1 + (-2) \times \frac{3}{2} + 0 \times 2}{\sqrt{3^2 + (-2)^2 + 0^2}\sqrt{1^2 + \left(\frac{3}{2}\right)^2 + 2^2}} = \frac{3 - 3}{\sqrt{13}\sqrt{\frac{29}{4}}} = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$

Q.2. Find the angle between the pair of lines given by

$$\vec{r} = 3i + 2j - 4k + \lambda(i + 2j + 2k) \text{and } \vec{r} = 5i - 2j + \mu(3i + 2j + 6k)$$

Ans.

Here, $\overrightarrow{b_1} = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\overrightarrow{b_2} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

The angle qbetween the two lines is given by

$$\cos\theta = \left|\frac{\overrightarrow{b_1}.\overrightarrow{b_2}}{\left|\overrightarrow{b_1}\right|\left|\overrightarrow{b_2}\right|}\right| = \left|\frac{(\hat{i}+2\hat{j}+2\hat{k}).(3\hat{i}+2\hat{j}+6\hat{k})}{\sqrt{1+4+4}\sqrt{9+4+36}}\right| = \left|\frac{3+4+12}{3\times7}\right| = \frac{19}{21}$$

Hence, $\theta = \cos^1\left(\frac{19}{21}\right)$

Q.3. Find the equation of line passing through the point (2, 1, 3) having the direction ratios 1, 1, – 2.

Ans.

Let the point $A(2, 1, 3)$ and $a = 1$, $b = 1$, and $c = -2$. Since, we know that cartesian equation of straight line passing through the point (x_1, y_1, z_1) having direction ratios, a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Therefore, equation of required line is

$$\frac{x-2}{1} = \frac{y-1}{1} = \frac{z-3}{-2}.$$

Q.4. Cartesian equation of a line *AB* is

$$\frac{2x-1}{2} = \frac{4-y}{7} = \frac{z+1}{2}$$

Write the direction ratios of a line parallel to *AB*.

Ans.

We have equations of line

$$\frac{2x-1}{2} = \frac{4-y}{7} = \frac{z-1}{2} \quad \Rightarrow \quad \frac{x-\frac{1}{2}}{1} = \frac{y-4}{-7} = \frac{z-(-1)}{2}$$

Direction ratios of given line are 1, −7, 2.

Hence, direction ratios of any parallel line are 1, −7, 2 or any multiples of ratios.

Q.5. Find the equation of a plane that cuts the coordinates axes at (*a*, 0, 0), (0, *b*, 0) and (0, 0, *c*).

Ans.

The equation of such plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Q.6. If a line makes an angle 30°, 60°, 90° with the positive direction of *x, y, z* axes respectively, then find its direction cosines.

Ans.

Since, the direction cosine of line which makes an angle of a, b, g with *x, y* and *z* are cos a, cos b and cos g.

∴ Dc's of given line are cos 30°, cos 60°, cos 90°

i.e., $\frac{\sqrt{3}}{2}, \frac{1}{2}, 0$

Short Answer Questions-I (PYQ)

[2 Mark]

Q.1. Write the vector equation of the following line.

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{6-z}{2}$$

Ans.

Cartesian form of the line is given as

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{-2}$$

The standard form of line's equation

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

We get by comparing that the given line passes through the point (x_1, y_1, z_1) *i.e.*, $(5, -4, 6)$ and direction ratios are (a, b, c) *i.e.*, $(3, 7, -2)$.

Now, we can write vector equation of line as

$$\vec{a} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$$

Q.2. Find the direction cosines of the line passing through two points (–2, 4, –5) and (1, 2, 3).

Ans.

We know that direction cosines of the line passing through two points $P\ (x_1, y_1, z_1)$ and Q (x_2, y_2, z_2) are given by

$$\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}, \quad \text{where } PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Here P is $(-2, 4, -5)$ and Q is $(1, 2, 3)$.

So $PQ = \sqrt{(1-(-2))^2 + (2-4)^2 + (3-(-5))^2} = \sqrt{77}$

Thus, the direction cosines of the line joining two points are $\frac{1+2}{\sqrt{77}}, \frac{2-4}{\sqrt{77}}, \frac{3+5}{\sqrt{77}}$

or $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$

Q.3. Find the value of l so that the lines $\frac{1-x}{3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$ **and** $\frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{6-z}{7}$ **are perpendicular to each other.**

Ans.

The given lines can be expressed as

$$\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2} \text{ and } \frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{z-6}{-7}$$

The direction ratios of these lines are –3, 2l, 2 and 3l, 1, –7 respectively.

Since the lines are perpendicular, therefore

$$-3(3\lambda)+(2\lambda(1)+2(-7)=0 \Rightarrow \quad -9\lambda+2\lambda-14=0$$

$$\Rightarrow \; -7\lambda=14 \Rightarrow \quad \lambda=-2$$

Short Answer Questions-I (OIQ)

[2 Mark]

Q.1. Show that the points *A* (2, 3, – 4), *B* (1, –2, 3) and *C* (3, 8, –11) are collinear.

Ans.

Direction ratios of the line joining *A* and *B* are (1 – 2), (–2 –3), (3 + 4) *i.e.*, –1, –5, 7

Direction ratios of the line joining *A* and *C* are (3 – 2), (8 – 3), (–11 + 4) *i.e.*, 1, 5, –7

It is clear that direction ratios of line *AB* and *AC* are proportional.

⇒ Parallel vectors of both lines are parallel to each other.

⇒ Both given lines are parallel.

⇒ But point A is common. So, points *ABC* are collinear.

Q.2. Find the equation of line in vector and cartesian form that passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ **and is in the direction** $\hat{i}+2\hat{j}-\hat{k}$

Ans.

P.V. of a point of line = $2\hat{i} - \hat{j} + 4\hat{k}$.

⇒the coordinate of that point = $(2, -1, 4)$

Therefore vector equation of the plane is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$$\vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$$

Also its cartesian form is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Here, $x_1 = 2, y_1 = -1, z_1 = 4$ and $a = 1, b = 2, c = -1$,

Therefore required equation in cartesian form is

$$\frac{x - 2}{1} = \frac{y+1}{2} = \frac{z - 4}{-1}$$

Q.5. Find the distance of the point whose position vector is $(4\hat{i} + 3\hat{j} - \hat{k})$ from the plane $\vec{r}(\hat{i} - 2\hat{j} + 3\hat{k}) = 4$.

Ans.

Given P.V. of point = $4\hat{i} + 3\hat{j} - \hat{k}$

$\Rightarrow$ Coordinates of point = $(4, 3, -1)$

Equation of plane $\vec{r}\,(\hat{i} - 2\hat{j} + 3\hat{k}) = 4$

$\therefore$ Cartesian form of equation

$x - 2y + 3z - 4 = 0$

If d be the distance of point from the given plane then.

$$d = \left|\frac{4\times1+3\times(-2)+(-1)\times3-4}{\sqrt{(1)^2+(-2)^2+3^2}}\right|$$

$$\Rightarrow\ d = \left|\frac{4-6-3-4}{\sqrt{14}}\right|$$

$$\Rightarrow\ d = \frac{9}{\sqrt{14}}\text{ units}$$

Long Answer Questions-I (PYQ)

[4 Mark]

Q.1. Find the shortest distance between the lines whose vector equations are:

$\vec{r} = (\hat{i}+\hat{j})+\lambda(2\hat{i}-\hat{j}+\hat{k})$ and $\vec{r} = (2\hat{i}+\hat{j}-\hat{k})+\mu(3\hat{i}-5\hat{j}+2\hat{k})$.

Ans.

Comparing the given equations with equations

$\vec{r} = \vec{a_1}+\lambda\vec{b_1}$ and $\vec{r} = \vec{a_2}+\mu\vec{b_2}$

We get $\vec{a_1} = \hat{i}+\hat{j}$, $\vec{b_1} = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{a_2} = 2\hat{i}+\hat{j}-\hat{k}$, $\vec{b_2} = 3\hat{i}-5\hat{j}+2\hat{k}$

Therefore, $\vec{a_2}-\vec{a_1} = (\hat{i}-\hat{k})$ and

$$\vec{b_1}\times\vec{b_2} = (2\hat{i}-\hat{j}+\hat{k})\times(3\hat{i}-5\hat{j}+2\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i}-\hat{j}-7\hat{k}$$

$$\left|\vec{b_1}\times\vec{b_2}\right| = \sqrt{9+1+49} = \sqrt{59}$$

Hence, the shortest distance between the given lines is given by

$$d = \left|\frac{(\vec{b_1}\times\vec{b_2}).(\vec{a_2}-\vec{a_1})}{\left|\vec{b_1}\times\vec{b_2}\right|}\right|$$

$$= \left|\frac{3-0+7}{\sqrt{59}}\right| = \frac{10}{\sqrt{59}} \text{ units.}$$

Q.2. Find the vector and cartesian equations of the line passing through the point *P*(3, 0, 1) and parallel to the planes $\vec{r}.(\hat{i}+2\hat{j}) = 0$ and $\vec{r}.(3\hat{j}-\hat{k}) = 0$.

Ans.

Let $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ be normal vector of given plane

$\vec{r}\,.\,(\hat{i}+2\hat{j})=0$ ...(i) and = $\vec{r}\,.\,(3\,\hat{j}-\hat{k})\,0$ ...(ii)

$\therefore\ \overrightarrow{n_1}=\hat{i}+2\hat{j}$ and $\overrightarrow{n_2}=3\hat{j}-\hat{k}$

since, required line is parallel to the plane (i) and (ii)

$\overrightarrow{n_1}\times\overrightarrow{n_2}$ is parallel vector of required line.

Now, $\overrightarrow{n_1}\times\overrightarrow{n_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 0\\ 0 & 3 & -1\end{vmatrix}=(-2-0)\hat{i}-(-1-0)\hat{j}+(3-0)\hat{k}=-2\hat{i}+\hat{j}+3\hat{k}$

Hence, the vector equation of required line is

$\vec{r}=(3\hat{i}+\hat{k})+\lambda(-2\hat{i}+\hat{j}+3\hat{k})$

Corresponding cartesian equation is

$\frac{x-3}{-2}=\frac{y}{1}=\frac{z-1}{3}$

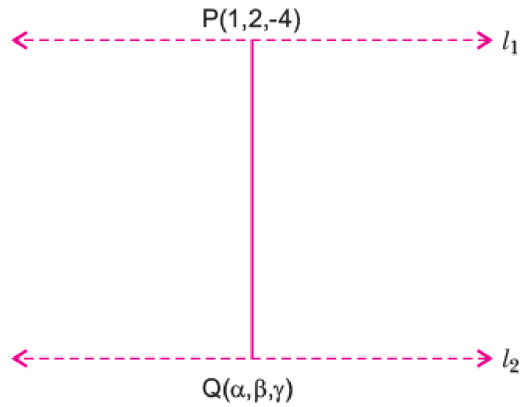
Q.3. Find the distance between the lines l_1 and l_2 given by

$l_1:\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k});\ \ l_2:\vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu(4\hat{i}+6\hat{j}+12\hat{k})$

Ans.

Given lines are

$l_1:\ \vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$



l_2: $\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$

After observation, we get $l_1 \parallel l_2$

Therefore, it is sufficient to find the perpendicular distance of a point of line l_1 to line l_2.

The coordinate of a point of l_1 is $P(1, 2, -4)$

Also the cartesian form of line l_2 is

$\frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}$ ... (i)

Let $Q(\alpha, \beta, \gamma)$ be foot of perpendicular drawn from P to line l_2

$\because$ Q(a, b, g) lie on line l_2

$\therefore \frac{\alpha-3}{4} = \frac{\beta-3}{6} = \frac{\gamma+5}{12} = \lambda$ (say)

$\Rightarrow a = 4\lambda + 3$, b = $6\lambda + 3$, $\gamma = 12\lambda - 5$

Again, $\because$ $\overrightarrow{PQ}$ is perpendicular to line l_2.

$\Rightarrow$ $\overrightarrow{PQ} \cdot \vec{b} = 0$, where $\vec{b}$ is parallel vector of l_2

$\Rightarrow(\alpha-1).4+(\beta-2).6+(\gamma+4).\ 12=0\ \left[\because\ \overrightarrow{PQ}=(\alpha-1)\hat{i}+(\beta-2)\hat{j}+(\gamma+4)\hat{k}\ \right]$

$\Rightarrow 4\alpha-4+6b-12+12g+48=0$

$\Rightarrow 4\alpha+6b+12g+32=0$

$\Rightarrow 4\ (4\lambda+3)+6\ (6\lambda+3)+12(12\lambda-5)+32=0$

$\Rightarrow 16\lambda+12+36\lambda+18+144\lambda-60+32=0$

$\Rightarrow 196\lambda+2=0\ \Rightarrow\ \lambda=\frac{-2}{196}=\frac{-1}{98}$

Coordinate of $Q\ \equiv\ \left(4\times(-\ \frac{1}{98})+3,\ 6\times\ (-\ \frac{1}{98})+3,\ 12\times(-\ \frac{1}{98})-\ 5\right)$

$\equiv\ \left(-\ \frac{2}{49}+3,-\ \frac{3}{49}+3,-\ \frac{6}{49}-\ 5\right)\equiv\left(\frac{145}{49},\ \frac{144}{49},-\ \frac{251}{49}\right)$

Therefore required perpendicular distance is

$$\sqrt{\left(\frac{145}{49}-1\right)^2+\left(\frac{144}{49}-2\right)^2+\left(\frac{-251}{49}+4\right)^2}=\sqrt{\left(\frac{96}{49}\right)^2+\left(\frac{46}{49}\right)^2+\left(\frac{-55}{49}\right)^2}$$

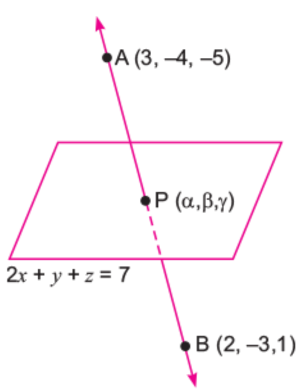
$$=\sqrt{\frac{96^2+46^2+55^2}{49^2}}=\sqrt{\frac{9216+2116+3025}{49^2}}$$

$$=\frac{\sqrt{14357}}{49}=\frac{7\sqrt{293}}{49}=\frac{\sqrt{293}}{7}\text{ units}$$

Q.4. Find the coordinates of the point where the line through the points (3, –4, –5) and (2, –3, 1) crosses the plane 2*x*+ *y* + *z* = 7.

Ans.

The equation of line passing through the points (3, – 4, – 5) and (2, – 3, 1) is



$$\frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5} \Rightarrow \frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6} \ldots(i)$$

Let the line (i) crosses at point $P(\alpha, \beta, \gamma)$ the plane $2x + y + z = 7 \ldots(ii)$

$\because$ P lies on line (i), therefore (a, b, g) satisfy equation (i)

$$\therefore \frac{\alpha-3}{-1}=\frac{\beta+4}{1}=\frac{\gamma+5}{6}=\lambda \text{ (say)}$$

$\alpha = -\lambda + 3; \beta = \lambda - 4$ and $\gamma = 6\lambda - 5$

Also $P(\alpha, \beta, \gamma)$ lie on plane (ii)

$\therefore 2\alpha + \beta + \gamma = 7$

$\Rightarrow 2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) = 7$

$\Rightarrow -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 = 7$

$\Rightarrow 5\lambda = 10 \quad \Rightarrow \quad \lambda = 2$

Hence, the coordinate of required point P is $(-2 + 3, 2 - 4, 6 \times 2 - 5)$ *i.e.*, $(1, -2, 7)$

Q.5. A line passes through (2, –1, 3) and is perpendicular to the lines $\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$. Obtain its equation in vector and cartesian form.

Ans.

Let $\overrightarrow{b}$ be parallel vector of required line.

$\Rightarrow \overrightarrow{b}$ is perpendicular to both given line.

$\Rightarrow \overrightarrow{b} = (2\hat{i} - 2\hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} + 2\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = (-4 - 2)\hat{i} - (4 - 1)\hat{j} + (4 + 2)\hat{k} = -6\hat{i} - 3\hat{j} + 6\hat{k}.$$

Hence, the equation of line in vector form is

$\overrightarrow{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(-6\hat{i} - 3\hat{j} + 6\hat{k}) \qquad \overrightarrow{r} = (2\hat{i} - \hat{j} + 3\hat{k}) - 3\lambda(2\hat{i} + \hat{j} - 2\hat{k})$

$\overrightarrow{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \mu(2\hat{i} + \hat{j} - 2\hat{k})$ $[\mu = -3\lambda]$

Equation in cartesian form is

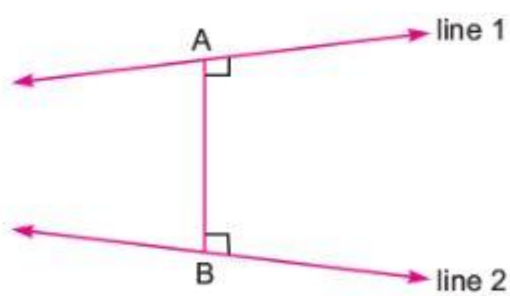
$$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-3}{-2}$$

Q.6. Find the shortest distance between the following lines :

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad \text{and} \quad \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

Ans.

Let $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = \lambda$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} = k$



Now, let's take a point on first line as

$A\ (\lambda + 3, -2\lambda + 5, \lambda + 7)$ and let

$B\ (7k - 1, -6k - 1, k - 1)$ be point on the second line

The direction ratio of the line AB

$7k - \lambda - 4, -6k + 2\lambda - 6, k - \lambda - 8$

Now, as AB is the shortest distance between line 1 and line 2 so,

$$(7k - \lambda - 4) \times 1 + (-6k + 2\lambda - 6) \times (-2) + (k - \lambda - 8) \times 1 = 0 \qquad ...(i)$$

and $(7k - \lambda - 4) \times 7 + (-6k + 2\lambda - 6) \times (-6) + (k - \lambda - 8) \times 1 = 0 \quad ...(ii)$

Solving equation (i) and (ii), we get

$\lambda = 0$ and $k = 0$

$\therefore\ A \equiv (3, 5, 7)$ and $B \equiv (-1, -1, -1)$

$\therefore\ AB = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = \sqrt{16 + 36 + 64} = \sqrt{116}$ units $= 2\sqrt{29}$ units

Q.7. Find the point on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance $3\sqrt{2}$ from the point (1, 2, 3).

Ans.

Let $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda$

$\therefore (3\lambda - 2, 2\lambda - 1, 2\lambda + 3)$ is any general point on the line

Now if the distance of the point from $(1, 2, 3)$ is $3\sqrt{2}$, then

$\Rightarrow \sqrt{(3\lambda - 2 - 1)^2 + (2\lambda - 1 - 2)^2 + (2\lambda + 3 - 3)^2} = (3\sqrt{2})$

$\Rightarrow (3\lambda - 3)^2 + (2\lambda - 3)^2 + 4\lambda^2 = 18 \Rightarrow 9\lambda^2 - 18\lambda + 9 + 4\lambda^2 - 12\lambda + 9 + 4\lambda^2 = 18$

$\Rightarrow 17\lambda^2 - 30\lambda = 0 \Rightarrow \lambda(17\lambda - 30) = 0$

$\Rightarrow \lambda = 0$ or $\lambda = \frac{30}{17}$

$\therefore$ Required point on the line is $(-2, -1, 3)$ or $\left(\frac{56}{17}, \frac{43}{17}, \frac{77}{17}\right)$

Q.8. Find the equation of the line passing through the point *P* (4, 6, 2) and the point of intersection of the line $\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{7}$ and the plane *x* + *y* – *z* = 8.

Ans.

Given line and plane are

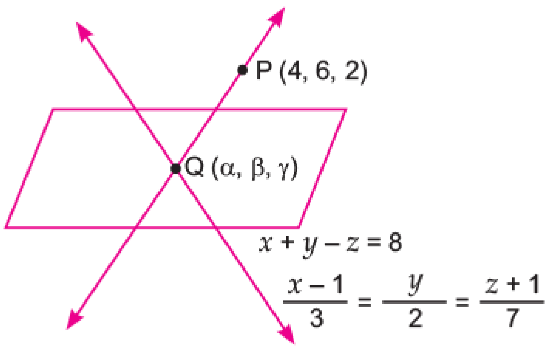
$\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{7} = \lambda \quad ...(i)$

and $x + y - z = 8 \; ...(ii)$

Let $Q(\alpha, \beta, \gamma)$ be point of intersection of equation (i) and (ii),

'Q' lie on line (i)

$\frac{\alpha - 1}{3} = \frac{\beta}{2} = \frac{\gamma + 1}{7} = \lambda$



$\Rightarrow \quad \alpha = 3\lambda + 1, b = 2\lambda, \gamma = 7\lambda - 1$

Also $Q\ (\alpha, \beta, \gamma)$ lie on plane (ii),

$\alpha + \beta - \gamma = 8$

$\Rightarrow 3\lambda + 1 + 2\lambda - 7\lambda + 1 = 8$

$\Rightarrow -2\lambda = 6 \Rightarrow \lambda = -3$

$\therefore$ Coordinates of $Q \equiv (-8, -6, -22)$.

Required equation of line passing through $P\ (4, 6, 2)$ and $Q\ (-8, -6, -22)$ is

$$\frac{x-4}{4+8} = \frac{y-6}{6+6} = \frac{z-2}{2+22}$$

$$\Rightarrow \frac{x-4}{12} = \frac{y-6}{12} = \frac{z-2}{24} \qquad \Rightarrow \qquad x-4 = y-6 = \frac{z-2}{2}$$

Q.9. Find the shortest distance between the following two lines:

$$\overrightarrow{r} = (1+\lambda)\hat{i} + (2-\lambda)\hat{j} + (\lambda+1)\hat{k} \text{ and } \overrightarrow{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu\ (2\hat{i} + \hat{j} + 2\hat{k})$$

Ans.

The given equation of the lines can be rearranged as given below.

$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

Thus, $\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b_1} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$, $\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$

Shortest distance between lines = $\left|\frac{(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}\right|$

We have $\vec{a_2} - \vec{a_1} = \hat{i} - 3\hat{j} - 2\hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -3\hat{i} + 0\hat{j} + 3\hat{k}$$

$$\left|\vec{b_1} \times \vec{b_2}\right| = \sqrt{9+9} = 3\sqrt{2}$$

$\therefore$ Shortest distance $\left|\frac{(\hat{i} - 3\hat{j} - 2\hat{k}).(-3\hat{i} + 3\hat{k})}{3\sqrt{2}}\right| = \left|\frac{-3-6}{3\sqrt{2}}\right| = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$ units.

Q.10. Find the vector and cartesian equations of the line passing through the point (2, 1, 3) and perpendicular to the lines
$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ **and** $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$ **.**

Ans.

Let the cartesian equation of the line passing through $(2, 1, 3)$ be

$$\frac{x-2}{a} = \frac{y-1}{b} = \frac{z-3}{c} \quad \ldots(i)$$

Since, line (i) is perpendicular to given line

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} \quad \ldots(ii)$$

and $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5} \quad \ldots(iii)$

$\therefore \; a + 2b + 3c = 0 \quad \ldots(iv)$

$-3a + 2b + 5c = 0 \quad \ldots(v)$

From equation (iv) and (v).

$$\frac{a}{10-6} = \frac{b}{-9-5} = \frac{c}{2+6} \Rightarrow \qquad \frac{a}{4} = \frac{b}{-14} = \frac{c}{8} = \lambda \quad \text{(say)}$$

$\Rightarrow \; a = 4\lambda, \, b = -14\lambda, \, c = 8\lambda$

Putting the value of a, b and c in (i), we get

$$\frac{x-2}{4\lambda} = \frac{y-1}{-14\lambda} = \frac{z-3}{8\lambda} \Rightarrow \qquad \frac{x-2}{4} = \frac{y-1}{-14} = \frac{z-3}{8}$$

$\Rightarrow \frac{x-2}{2} = \frac{y-1}{-7} = \frac{z-3}{4}$, which is the cartesian form

The vector form is $\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k})$

Q.11. Find the angle between the line $\frac{x+1}{2} = \frac{3y+5}{9} = \frac{3-z}{-6}$ and the plane $10x + 2y - 11z = 3$.

Ans.

Given line can be rearranged to get

$$\frac{x-(-1)}{2}=\frac{y-(-5/3)}{3}=\frac{z-3}{6}$$

Its direction ratios are 2, 3, 6.

Direction ratios of normal to the plane $10x + 2y - 11z = 3$ are 10, 2, – 11

Angle between the line and plane

$$\sin\theta=\frac{2\times10+3\times2+6(-11)}{\sqrt{4+9+36}\ \sqrt{100+4+121}}=\frac{20+6-66}{7\times15}=\frac{-40}{105}$$

$$\sin\theta=\frac{-8}{21}\quad \text{or}\quad \theta=\sin^{-1}\left(\frac{-8}{21}\right)$$

Q.12. Find the equation of the perpendicular drawn from the point (1, –2, 3) to the plane $2x - 3y + 4z + 9 = 0$. Also, find the coordinates of the foot of the perpendicular.

Ans.

Let the foot of the perpendicular on the plane be *A*.

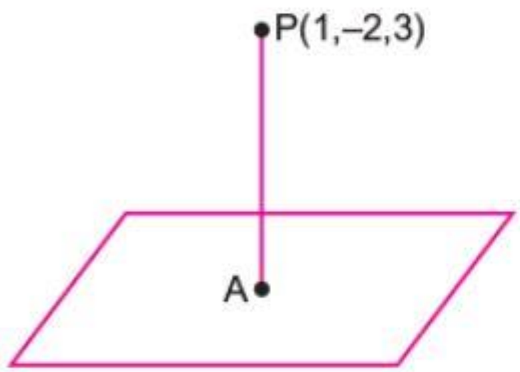
PA perpendicular to the plane

$2x - 3y + 4z + 9 = 0$

Dr's of *PA* = 2, –3, 4

Equation of *PA* can be written as

$$\frac{x-1}{2}=\frac{y+2}{-3}=\frac{z-3}{4}=\lambda$$



General points of line $PA = (2\lambda + 1, -3\lambda - 2, 4\lambda + 3)$

The point is on the plane hence

$2(2\lambda + 1) - 3(-3\lambda - 2) + 4(4\lambda + 3) + 9 = 0$

$\Rightarrow 29\lambda + 29 = 0$ or $\lambda = -1$

$\therefore$Coordinates of foot of perpendicular are $(-1, 1, -1)$.

Q.13. Find the cartesian equation of the plane passing through the points *A* (0, 0, 0) and *B* (3, –1, 2) and parallel to the line $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$.

Ans.

Equation of plane is given by

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Given plane passes through $(0, 0, 0)$

$\therefore\ a(x - 0) + b(y - 0) + c(z - 0) = 0 \ldots(i)$

Plane (i), passes through $(3, -1, 2)$

$\therefore\ 3a - b + 2c = 0 \ldots(ii)$

Also, plane (i) is parallel to the line

$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$

$a - 4b + 7c = 0 \qquad \ldots(iii)$

Eliminating a, b, c from equations (i), (ii) and (iii), we get

$$\begin{vmatrix} x & y & z \\ 3 & -1 & 2 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \quad x\begin{vmatrix} -1 & 2 \\ -4 & 7 \end{vmatrix} - y\begin{vmatrix} 3 & 2 \\ 1 & 7 \end{vmatrix} + z\begin{vmatrix} 3 & -1 \\ 1 & -4 \end{vmatrix} = 0$$

$\Rightarrow x(-7 + 8) - y(21 - 2) + z(-12 + 1) = 0$

$\Rightarrow x - 19y - 11z = 0$, which is the required equation

Q.14. Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance 5 units from the point *P* (1, 3, 3).

Ans.

Given cartesian form of line is:

$$\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}=\mu$$

$\therefore$ General point on line is $(3\mu-2, 2\mu-1, 2\mu+3)$

Since, distance of point on line from $P(1, 3, 3)$ is 5 units.

$$\therefore \sqrt{(3\mu-2-1)^2+(2\mu-1-3)^2+(2\mu+3-3)^2}=5$$

$$\Rightarrow (3\mu-3)^2+(2\mu-4)^2+(2\mu)^2=25$$

$$\Rightarrow 17\mu^2-34\mu=0$$

$$\Rightarrow 17\mu(\mu-2)=0 \Rightarrow \mu=0, 2$$

$\therefore$ Required point on line is $(-2, -1, 3)$ for $\mu=0$, or $(4, 3, 7)$ for $\mu=2$.

Q.15. Find the distance of the point *P* (6, 5, 9) from the plane determined by the points *A* (3, –1, 2), *B* (5, 2, 4) and *C*(–1, –1, 6).

Ans.

Plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$ is

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 5-3 & 2+1 & 4-2 \\ -1-3 & -1+1 & 6-2 \end{vmatrix}=0 \quad \Rightarrow \quad \begin{vmatrix} x-3 & y+1 & x-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix}=0$$

$$(x-3)\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix}-(y+1)\begin{vmatrix} 2 & 2 \\ -4 & 4 \end{vmatrix}+(z-2)\begin{vmatrix} 2 & 3 \\ -4 & 0 \end{vmatrix}=0$$

$$\Rightarrow 12x-36-16y-16+12z-24=0$$

$$\Rightarrow 3x-4y+3z-19=0$$

Distance of this plane from point $P(6, 5, 9)$ is

$$\left|\frac{(3\times6)-(4\times5)+(3\times9)-19}{\sqrt{(3)^2+(4)^2+(3)^2}}\right|=\left|\frac{18-20+27-19}{\sqrt{9+16+9}}\right|=\frac{6}{\sqrt{34}} \text{ units.}$$

Q.16. Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ **and** $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ **intersect. Also find their point of intersection.**

Ans.

Given lines are

$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}\quad \ldots(i)$

$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}\quad \ldots(ii)$

Let two lines (i) and (ii) intersect at a point $P(\text{a, b}, \gamma)$.

$\Rightarrow$ (α, β, γ) satisfy line (i)

$\Rightarrow \frac{\alpha+1}{3}=\frac{\beta+3}{5}=\frac{\gamma+5}{7}=\lambda$ (say)

$\Rightarrow a = 3\lambda - 1,\ b = 5\lambda - 3,\ y = 7\lambda - 5\ , \ldots(iii)$

Again (α, β, γ) also lie on (ii)

$\frac{\alpha-2}{1}=\frac{\beta-4}{3}=\frac{\gamma-6}{5} \quad \Rightarrow \quad \frac{3\lambda-1-2}{1}=\frac{5\lambda-3-4}{3}=\frac{7\lambda-5-6}{5}$

$\Rightarrow \frac{3\lambda-3}{1}=\frac{5\lambda-7}{3}=\frac{7\lambda-11}{5}$

I II III

From I and II

$\frac{3\lambda-3}{1}=\frac{5\lambda-7}{3}$

From II and III

$\frac{5\lambda-7}{3}=\frac{7\lambda-11}{5}$

$\Rightarrow 9\lambda - 9 = 5\lambda - 7$ $\Rightarrow \quad 25\lambda - 35 = 21\lambda - 33$

$\Rightarrow 4\lambda = 2$ $\Rightarrow 4\lambda = 2$

$\Rightarrow \ \lambda = \frac{1}{2} \quad \Rightarrow \quad \lambda = \frac{1}{2}$

Since, the value of l in both the cases is same

$\Rightarrow$ Both lines intersect each other at a point.

$\therefore$Intersecting point $= (\alpha, \beta, \gamma) = \left(\frac{3}{2} - 1,\ \frac{5}{2} - 3,\ \frac{7}{2} - 5\right)$ [From (iii)]

$\left(\frac{1}{2}, -\ \frac{1}{2}, \frac{-3}{2}\right)$

Q.17. Find the vector and cartesian equations of the line passing through the point (1, 2, –4) and perpendicular to the two lines

$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ **and** $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

OR

Find the equation of a line passing through the point (1, 2, –4) and perpendicular to two lines

$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$ **and** $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

Ans.

Let the cartesian equation of line passing through $(1, 2, -4)$ be

$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c} \quad ...(i)$

Given lines are

$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad ...(ii)$

$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad ...(iii)$

Obviously parallel vectors $\vec{b_1}$, $\vec{b_2}$ and $\vec{b_3}$ of (i), (ii) and (iii) respectively are given as

$\vec{b_1} = a\hat{i} + b\hat{j} + c\hat{k}; \; \vec{b_2} = 3\hat{i} - 16\hat{j} + 7\hat{k}; \; \vec{b_3} = 3\hat{i} + 8\hat{j} - 5\hat{k}$

According to question

$(i) \perp (ii) \quad \vec{b_1} \perp \vec{b_2} \Rightarrow \quad \Rightarrow \vec{b_1}.\vec{b_2} = 0$

$(i) \perp (iii) \quad \Rightarrow \vec{b_1} \perp \vec{b_3} \quad \Rightarrow \vec{b_1}.\,\vec{b_3} = 0$

Hence, $3a - 16b + 7c = 0 \quad ...(iv)$

and $3a + 8b - 5c = 0 \quad ...(v)$

From equation (iv) and (v), we get

$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$

$\Rightarrow \frac{a}{24} = \frac{b}{36} = \frac{c}{72} \quad \Rightarrow \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{6} = \lambda$ (say)

$\Rightarrow a = 2\lambda,\ b = 3\lambda,\ c = 6\lambda$

Putting the value of a, b, c in (i), we get the required cartesian equation of line as

$\frac{x-1}{2\lambda} = \frac{y-2}{3\lambda} = \frac{z+4}{6\lambda} \quad \Rightarrow \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$

Hence, vector equation is

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

Q.18. Find the angle between the following pair of lines:

$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$ **and** $\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$

and check whether the lines are parallel or perpendicular.

Ans.

The equation of given lines can be written in standard form as

$$\frac{x-2}{2}=\frac{y-1}{7}=\frac{z-(-3)}{-3} \quad \ldots(i)$$

and $\frac{x-(-2)}{-1}=\frac{y-4}{2}=\frac{z-5}{4} \ldots(ii)$

If $\vec{b_1}$ and $\vec{b_2}$ are vectors parallel to lines (i) and (ii) respectively, then

$\vec{b_1}=2\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{b_2}=-\hat{i}+2\hat{j}+4\hat{k}$

Obviously, if q is the angle between lines (i) and (ii) then q is also the angle between $\vec{b_1}$ and $\vec{b_2}$.

$$\therefore \cos\theta=\left|\frac{\vec{b_1}\cdot\vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}\right|=\left|\frac{(2i+7j-3k).(-i+2j+4k)}{\sqrt{2^2+7^2+(-3)^2}.\sqrt{(-1)^2+2^2+4^2}}\right|=\left|\frac{-2+14-12}{\sqrt{62}.\sqrt{21}}\right|=0$$

$$\Rightarrow \quad \theta=\frac{\pi}{2}$$

Angle between both lines is 90°.

Hence, given lines are perpendicular to each other.

Q.19. A line passing through the point *A* with position vector $\vec{a}=4\hat{i}+2\hat{j}+2\hat{k}$ **is parallel to the vector** $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}.$ **Find the length of the perpendicular drawn on this line from a point *P* with position vector** $\vec{r_1}=\hat{i}+2\hat{j}+3\hat{k}.$

Ans.

The equation of line passing through the point A and parallel to $\vec{b}$ is given in cartesian form as

$$\frac{x-4}{2}=\frac{y-2}{3}=\frac{z-2}{6} \quad \ldots(i)$$

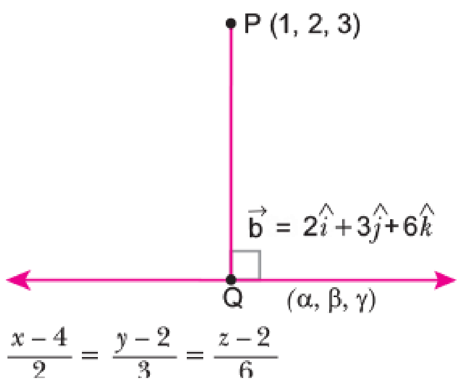
Let $Q(\alpha, \beta, \gamma)$ be foot of perpendicular drawn from point P to the line (i).

Co-ordinate or point $P \equiv (1, 2, 3)$ $[\because P.V.$ of P is $\hat{i}+2\hat{j}+3\hat{k}]$

Since, Q lie on line (i)

$$\frac{\alpha-4}{2}=\frac{\beta-2}{3}=\frac{\gamma-2}{6}=\lambda$$

$\alpha=2\lambda+4,\ \beta=3\lambda+2,\ \gamma=6\lambda+2.$



Now, $\overrightarrow{PQ}=(\alpha-1)\hat{i}+(\beta-2)\hat{j}+(\gamma-3)\hat{k}$

Obviously, $\overrightarrow{PQ} \perp \vec{b}$ $\quad \therefore \quad$ $\overrightarrow{PQ} \cdot \vec{b}=0$

$\Rightarrow \quad 2(\alpha-1)+3(\beta-2)+6(\gamma-3)=0$

$\Rightarrow \quad 2\alpha-2+3\beta-6+6\gamma-18=0$

$\Rightarrow \quad 2\alpha+3\beta+6\gamma-26=0$

Putting the value of α, β, γ; we get

$2(2\lambda+4)+3(3\lambda+2)+6(6\lambda+2)-26=0$

$\Rightarrow \quad 4\lambda + 8 + 9\lambda + 6 + 36\lambda + 12 - \ 26 = 0$

$\Rightarrow \quad 49\lambda \ = \ 0 \qquad \Rightarrow \qquad \lambda = 0$

Hence, the co-ordinate of $Q \equiv (4, 2, 2)$

$\therefore$ Length of perpendicular $Q \equiv (4, 2, 2)$

$= \sqrt{9 + 0 + 1} = \sqrt{10}$ nits.

Q.20. Find the coordinates of the point, where the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ **intersects the plane *x* – *y* + *z* – 5 = 0. Also find the angle between the line and the plane.**

Ans.

Let the given line

$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} \quad \ldots(i)$

intersect at point P (a, b, g) to the plane $x - y + z - 5 = 0 \quad \ldots(ii)$

$\because \quad P(\alpha, \beta, \gamma)$ lie on line (i)

$\therefore \frac{\alpha - 2}{3} = \frac{\beta + 1}{4} = \frac{\gamma - 2}{2} = \lambda$ (say)

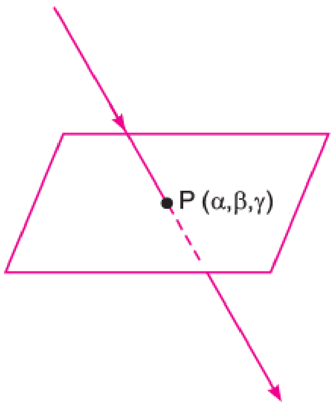
$\alpha = 3\lambda + 2$; b = $4\lambda - 1$; g = $2\lambda + 2$

Also, $P(\alpha, \beta, \gamma)$ lies on plane (ii)

$\therefore (3\lambda + 2) - (4\lambda - 1) + (2l + 2) - \ 5 = 0$

$\Rightarrow 3\lambda + 2 - 4\lambda + 1 + 2\lambda + 2 - 5 = 0 \Rightarrow \lambda = 0$

$\therefore \alpha = 2, \beta = -1, \gamma = 2$



Hence, co-ordinate of required point = (2, – 1, 2)

Now, find angle between line (*i*) and plane (*ii*)

If θ be the required angle, then

$$\sin\theta = \left|\frac{\vec{b}.\vec{n}}{|\vec{b}|.|\vec{n}|}\right|$$

$$\therefore\ \sin\theta = \left|\frac{1}{\sqrt{9+16+4}.\ \sqrt{1^2+(-1)^2+1^2}}\right| = \left|\frac{1}{\sqrt{29}.\ \sqrt{3}}\right| \qquad \left[\begin{array}{l}\because\ \vec{b} = 3\hat{i}+4\hat{j}+2\hat{k} \\ \quad \vec{n} = \hat{i}-\hat{j}+\hat{k} \\ \therefore\ \vec{b}\ .\ \vec{n} = 3-4+2=1\end{array}\right]$$

$$\sin\theta = \frac{1}{\sqrt{87}} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$$

Q.21. Find the coordinates of the point where the line through the points *A*(3, 4, 1) and *B*(5, 1, 6) crosses the *XZ* plane. Also find the angle which this line makes with the *XZ* plane.

Ans.

Let P (a, b, g) be the point at which the given line crosses the XZ plane.

Now the equation of given line AB is

$$\frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5}$$

Since $P(\alpha, \beta, \gamma)$ lies on line (i)

$$\therefore \quad \frac{\alpha-3}{2}=\frac{\beta-4}{-3}=\frac{\gamma-1}{5}=\lambda \text{ (say)}$$

$\Rightarrow \alpha = 2\lambda + 3$; b $= -3\lambda + 4$ and g $= 5\lambda + 1$

Also $P(\alpha, \beta, \gamma)$ lie on XZ plane, *i.e.,* $y = 0$ $(0x + 1y + 0z = 0)$

0a + 1. b + 0. g = 0

$\Rightarrow \beta = 0 \Rightarrow -3\lambda + 4 = 0 \Rightarrow \lambda = \frac{4}{3}$

Hence, the co-ordinates of required point P is

$$\alpha = 2\times\frac{4}{3}+3=\frac{8}{3}+3=\frac{17}{3}$$

$$\beta=-3\times\frac{4}{3}+4=0$$

$$\gamma=5\times\frac{4}{3}+1=\frac{23}{3}$$

$\therefore$ Co-ordinate of required point is $\left(\frac{17}{3},0,\frac{23}{3}\right)$

Let q be the angle made by line AB with XZ plane.

$$\therefore \sin\theta = \left|\frac{\vec{n}\cdot\vec{b}}{|\vec{b}|\,|\vec{n}|}\right|$$

Here $\vec{n} = \hat{j}$

$$\vec{b} = 2\hat{i} - 3\hat{j} + 5\hat{k}$$

$$|\vec{n}| = 1 \text{ and } |\vec{b}| = \sqrt{4+9+25} = \sqrt{38}$$

$$\Rightarrow \sin\theta = \left|\frac{\hat{j}.\,(2\hat{i} - 3\hat{j}+5\hat{k})}{1.\,\sqrt{38}}\right| = \left|\frac{-3}{\sqrt{38}}\right|$$

$$\Rightarrow \sin\theta = \frac{3}{\sqrt{38}} \qquad \Rightarrow \qquad \theta = \sin^{-1}\left(\frac{3}{\sqrt{38}}\right)$$

Q.22. Find the vector equation of the line passing through the point *A*(1, 2, –1) and parallel to the line 5x – 25 = 14 – 7y = 35z.

Ans.

Given line is

$$5x - 25 = 14 - 7y = 35z$$

$$\Rightarrow \frac{x-5}{\frac{1}{5}} = \frac{2-y}{\frac{1}{7}} = \frac{z-0}{\frac{1}{35}}$$

$$\Rightarrow \frac{x-5}{\frac{1}{5}} = \frac{y-2}{-\frac{1}{7}} = \frac{z-0}{\frac{1}{35}}$$

$$= \frac{x-5}{7} = \frac{y-2}{-5} = \frac{z-0}{1} \qquad \ldots (i)$$

Hence, parallel vector of given line *i.e.,*

$\vec{b} = 7\hat{i} - 5\hat{j} + \hat{k}$

Since required line is parallel to given line (i)

$\Rightarrow \quad \vec{b} = 7\hat{i} - 5\hat{j} + \hat{k}$ will also be parallel vector of required line which passes through $A(1, 2, -1)$.

Therefore, required vector equation of line is

$\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k})$

Q.23. Find the co-ordinates of the point where the line $\vec{r} = (-\hat{i} - 2\hat{j} - 3\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 3\hat{k})$ **meets the plane which is perpendicular to the vector** $\vec{n} = \hat{i} + \hat{j} + 3\hat{k}$ **and at a distance of** $\frac{4}{\sqrt{11}}$ **from origin.**

Ans.

We know that the equation of plane is

$= \vec{r}\,.\,\hat{n}\ d;$ where is normal unit vector and d is perpendicular distance from origin.

Here, $\hat{n} = \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{1^2 + 1^2 + 3^2}} = \frac{1}{\sqrt{11}}(\hat{i} + \hat{j} + 3\hat{k})$ and $d = \frac{4}{\sqrt{11}}$

$\therefore$ Equation of plane

$\vec{r}\,.\frac{1}{\sqrt{11}}(\hat{i} + \hat{j} + 3\hat{k}) = \frac{4}{\sqrt{11}}$

$\Rightarrow \quad \vec{r}\,.(\hat{i} + \hat{j} + 3\hat{k}) = 4$

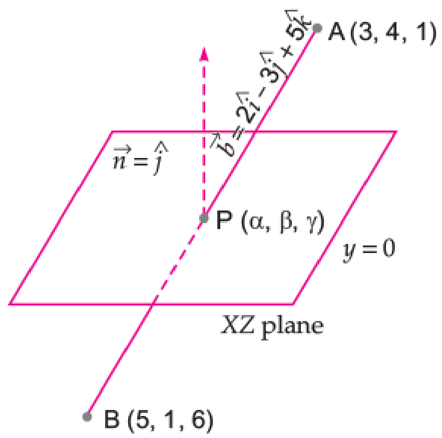
$\Rightarrow \quad x + y + 3z = 4 \quad ...(i)$

Equation of given line

$$\vec{r} = (-\hat{i} - 2\hat{j} - 3\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 3\hat{k})$$

Its cartesian form is

$$\frac{x+1}{3} = \frac{y+2}{4} = \frac{z+3}{3} \quad ...(ii)$$



Let Q (α, β, γ,) be the point of intersection of (i) & (ii)

$\because$ Q lies on (ii)

$$\therefore \quad \frac{\alpha+1}{3} = \frac{\beta+2}{4} = \frac{\gamma+3}{3} = \lambda$$

$\Rightarrow$ a = 3λ – 1, b = 4λ – 2, g = 3λ – 3

Also, Q lies on (i)

$\therefore \alpha + \beta + 3\gamma = 4$

$3\lambda - 1 + 4\lambda - 2 + 9\lambda - 9 = 4$

$\Rightarrow \; 16\lambda = 16 \;\Rightarrow\; \lambda = 1$

$\therefore \alpha = 2, \beta = 2, \gamma = 0$

$\therefore$ Required point of intersection $\equiv (2, 2, 0)$

Q.24. A variable plane which remains at a constant distance 3*p* from the origin cuts the coordinate axes at *A*, *B*, *C*. Show that the locus of the centroid of triangle *ABC* is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$.

Ans.

Let the given variable plane meets X, Y and Z axes at $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

Therefore the equation of given plane is given by

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \dots (i)$

Let (α, β, γ) be the coordinates of the centroid of triangle ABC. Then

$\alpha=\frac{a+0+0}{3}=\frac{a}{3} \quad \Rightarrow \quad a=3\alpha$

$\beta=\frac{0+b+0}{3}=\frac{b}{3} \quad \Rightarrow \quad b=3\beta$

$\gamma=\frac{0+0+c}{3}=\frac{c}{3} \quad \Rightarrow \quad c=3\gamma$

$\because \quad 3p$ is the distance from origin to the plane (i)

$$\Rightarrow \quad 3p=\frac{0.\frac{1}{a}+0.\frac{1}{b}+0.\frac{1}{c}-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

$$\Rightarrow \sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}=-\frac{1}{3p}$$

Squaring both sides, we have

$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{9p^2}$

$\Rightarrow \frac{1}{9\alpha^2}+\frac{1}{9\beta^2}+\frac{1}{9\gamma^2}=\frac{1}{9p^2}$ [Putting value of $a=3\alpha, b=3\beta, c=3\gamma$]

$\Rightarrow \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{1}{p^2}$

Therefore, Locus of (α,β,γ) is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$ Hence proved.

Q.25. Find the image P′ of the point *P* having position vector $\hat{i} + 3\hat{j} + 4\hat{k}$ in the plane $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) + 3 = 0.$ Hence find the length of PP′.

Ans.

Let given point be $P\,(1, 3, 4)$ and the equation of given plane in cartisan form be

$2x - y + z + 3 = 0 \qquad ...(i)$

Let $R\,(x_1, y_1, z_1)$ of foot of perpendicular and $Q\,(\alpha, \beta, \gamma)$ be the image of P

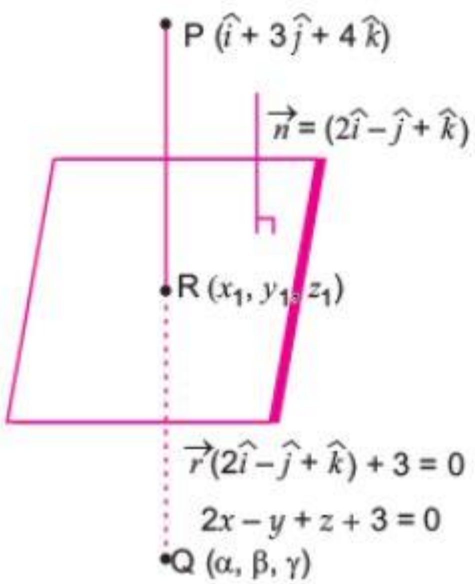
Since, $R\,(x_1, y_1, z_1)$ lie on plane (i)

$2x_1 - y_1 + z_1 + 3 = 0 \qquad ...(ii)$

Also, normal vector $\vec{n}$ of plane (i) is $\vec{n} = 2\hat{i} - \hat{j} + \hat{k}$

and $\overrightarrow{PR} = (x_1 - 1)\hat{i} + (y_1 - 3)\hat{j} + (z_1 - 4)\hat{k}$

$\therefore \; \overrightarrow{PR} \,||\, \vec{n}$



$\Rightarrow \quad \frac{x_1 - 1}{2} = \frac{y_1 - 3}{-1} = \frac{z_1 - 4}{1} = \lambda$

$\Rightarrow \quad x_1 = 2\lambda + 1, y_1 = -\lambda + 3, z_1 = \lambda + 4$

Putting x_1, y_1, z_1 in (*ii*) we get

$\Rightarrow \quad 2(2\lambda + 1) - (-\lambda + 3) + (\lambda + 4) + 3 = 0$

$\Rightarrow \quad 4\lambda + 2 + \lambda - 3 + \lambda + 4 + 3 = 0$

$\Rightarrow \quad 6\lambda + 6 = 0 \quad \Rightarrow \quad \lambda = -1$

$\therefore \quad R \equiv (x_1, y_1, z_1) \equiv (-1, 4, 3)$

Since R is the mid point of PQ

$\therefore \quad -1 = \frac{\alpha + 1}{2} \Rightarrow \alpha = -3$

$4 = \frac{\beta + 3}{2} \Rightarrow \beta = 5$

$3 = \frac{\gamma + 4}{2} \Rightarrow \gamma = 2$

Hence, image $Q = (-3, 5, 2)$

$$PP' = \sqrt{(-3-1)^2 + (5-3)^2 + (2-4)^2}$$
$$= \sqrt{16 + 4 + 4} = \sqrt{24}$$
$$= 2\sqrt{6} \text{ units}$$

Long Answer Questions-I (OIQ)

[4 Mark]

Q.1. Find the equation of the plane passing through the point (–1, 2, 1) and perpendicular to the line joining the points (–3, 1, 2) and (2, 3, 4). Also, find the perpendicular distance of the origin from this plane.

Ans.

Let the equation of plane passing through $(-1, 2, 1)$ be

$a(x + 1) + b(y - 2) + c(z - 1) = 0 \quad ...(i)$

Now the equation of line joining point $(-3, 1, 2)$ and $(2, 3, 4)$ is

$\frac{x+3}{2+3} = \frac{y-1}{3-1} = \frac{z-2}{4-2} \quad \Rightarrow \quad \frac{x+3}{5} = \frac{y-1}{2} = \frac{z-2}{2} \quad ...(ii)$

$\because$ Plane (i) is perpendicular to line (ii)

$\Rightarrow \frac{a}{5} = \frac{b}{2} = \frac{c}{2} = \lambda$ (say) $\quad$ [$\because$ Normal vector of (i) is parallel to line (ii)]

$\Rightarrow \quad a = 5\lambda, b = 2\lambda, c = 2\lambda$

Hence, equation of required plane is

$5\lambda\ (x+1)+2\lambda\ (y-2)+2\lambda\ (z-1) = 0$

$\Rightarrow 5x + 5 + 2y - 4 + 2z - 2 = 0$

$\Rightarrow 5x + 2y + 2z - 1 = 0$

If d is the distance from origin to plane then

$$d = \left|\frac{0.x+0.y+0.z-1}{\sqrt{5^2+2^2+2^2}}\right| = \frac{1}{\sqrt{33}}.$$

Q.2. Show that the line $\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 4\hat{k}).$ **is parallel to the plane** $\vec{r}.\ (\hat{i} + 5\hat{j} + \hat{k}) = 5$ **Also, find the distance between them.**

Ans.

As we know that the line $\vec{r} = \vec{a} + \lambda\ \vec{b}$ is parallel to the plane $\vec{r}\ .\ \vec{n} = q$ only when $\vec{b}\ .\ \vec{n} = 0$ and the distance between the line and the plane is given by $\frac{\left|\vec{a}.\vec{n} - q\right|}{\left|\vec{n}\right|}$.

Here, $\vec{a} = 2\hat{i} - 2\hat{j} + 3\hat{k},\quad \vec{b} = \hat{i} - \hat{j} + 4\hat{k},\quad \vec{n} = \hat{i} + 5\hat{j} + \hat{k}$ and $q = 5$.

Now, $\vec{b}.\vec{n} = (\hat{i} - \hat{j} + 4\hat{k}).\ (\hat{i} + 5\hat{j} + \hat{k})$

$= \{1 \times 1 + (-1) \times 5 + 4 \times 1\} = 0$

Hence, the given line is parallel to the given plane.

Now, distance between the given line and the given plane

$$= \frac{\left|\vec{a}.\vec{n} - q\right|}{\left|\vec{n}\right|} = \frac{\left|(2\hat{i} - 2\hat{j} + 3\hat{k}).\ (\hat{i} + 5\hat{j} + \hat{k}) - 5\right|}{\left|\hat{i} + 5\hat{j} + \hat{k}\right|}$$

$$= \frac{|2\times1+(-2)\times5+3\times1 - 5|}{\sqrt{(1)^2+(5)^2+(1)^2}} = \frac{|2 - 10+3 - 5|}{\sqrt{27}} = \frac{10}{3\sqrt{3}} \text{ units.}$$

Q.3. Find the vector equation of a line passing through the point with position vector $(2\hat{i} - 3\hat{j} - 5\hat{k})$ perpendicular to the plane $\vec{r}.\,(6\hat{i} - 3\hat{j} + 5\hat{k})$ +2=0. Also find the point of intersection of this line and the plane.

Ans.

As the required line is perpendicular to the plane

$\vec{r}.\,(6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0 \quad ...(i)$

So, the required line is parallel to $\vec{n} = 6\hat{i} - 3\hat{j} + 5\hat{k}$

Thus, the required line passes through the point with position vector $\vec{a} = 2\hat{i} - 3\hat{j} - 5\hat{k}$ and is parallel to $\vec{n} = 6\hat{i} - 3\hat{j} + 5\hat{k}$.

Hence, the vector equation of the required line is $\vec{r} = \vec{a} + \lambda\,\vec{n}$

i.e., $\vec{r} = (2\hat{i} - 3\hat{j} - 5\hat{k}) + \lambda\,(6\hat{i} - 3\hat{j} + 5\hat{k}) \;...(ii)$

If the line (ii) meets the plane (i), then

$[(2\hat{i} - 3\hat{j} - 5\hat{k}) + \lambda(6\hat{i} - 3\hat{j} + 5\hat{k})].\,(6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$

$\Rightarrow\; [(2 + 6\lambda)\hat{i} - (3 + 3\lambda)\hat{j} + (5\lambda - 5)\hat{k}].\,(6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$

$\Rightarrow 6\,(2+6l) + 3\,(3 + 3l) + 5\,(5l - 5) = -2$

$\Rightarrow 70l = 2 \Rightarrow\; \lambda = \frac{1}{35}$

Substituting $\lambda = \frac{1}{35}$ in (ii), we get

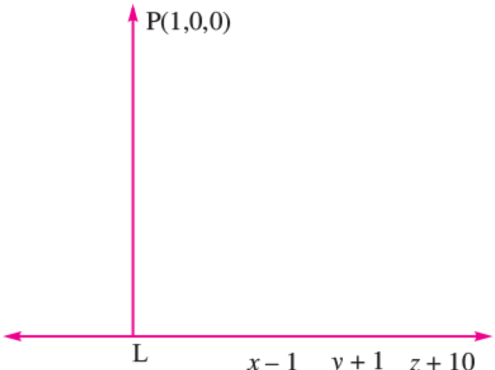
$\vec{r} = (2\hat{i} - 3\hat{j} - 5\hat{k}) + \frac{1}{35}(6\hat{i} - 3\hat{j} + 5\hat{k}) = \frac{1}{35}(76\hat{i} - 108\hat{j} - 170\hat{k})$

Hence, the required point of intersection is $\left(\frac{76}{35}, \frac{-108}{35}, \frac{-170}{35}\right)$ *i.e.*, $\left(\frac{76}{35}, \frac{-108}{35}, \frac{-34}{7}\right)$.

Q.4. Find the perpendicular distance of the point (1, 0, 0) from the line $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$.

Ans.

Let *L* be the foot of perpendicular drawn from the point *P* (1, 0, 0) on the given line.



$$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$$

Let $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}=\lambda$

The coordinates of general point on the line are

$x = 2\lambda + 1,\ y = -(3\lambda + 1),\ z = 8\lambda - 10$

Then the coordinates of L

becomes $[2\lambda + 1, -(3\lambda + 1), 8\lambda - 10]$

Therefore, direction ratios of PL are

$2\lambda, -(3\lambda + 1), (8\lambda - 10)$ respectively.

Direction ratios of the given line are 2, −3, 8.

Since $PL \perp$ given line, therefore,

$2.\ 2\lambda + 3(3\lambda + 1) + 8(8\lambda - 10) = 0$

$\Rightarrow 4\lambda + 9\lambda + 3 + 64\lambda - 80 = 0 \Rightarrow \lambda = 1$

Putting $\lambda = 1$ in $[2\lambda + 1, -(3\lambda + 1), 8\lambda - 10]$, we find that required foot of perpendicular is $[3, -4, -2]$.

$\therefore$ Length $PL = \sqrt{(3-1)^2 + (-4-0)^2 + (-2-0)^2}$

$= \sqrt{4+16+4} = \sqrt{24}$ units.

Q.5. The equation of motion of a point in space is $x = 2t$, $y = -4t$, $z = 4t$ where t measured in hour and the co-ordinates of moving point in kilometers. Find the distance of the point from the starting point *O* (0, 0, 0) in 10 hours.

Ans.

Eliminating 't' from the given equations, we get the equation of the path as given point

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{4} \quad \Rightarrow \quad \frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$$

Obviously, the path of the point is a straight line passing through origin $(0, 0, 0)$

When $t = 10$ hours the position of point is at $x = 20$, $y = -40$, $z = 40$.

i.e., After 10 hours the position of point will be at $(20, -40, 40)$

Therefore, required distance = distance between point $(0, 0, 0)$ and $(20, -40, 40)$

$$= \sqrt{(20-0)^2 + (-40-0)^2 + (40-0)^2}$$

$$= \sqrt{400 + 1600 + 1600}$$

$$= \sqrt{3600} = 60 \text{ km.}$$

Long Answer Questions-II (PYQ)

[6 Mark]

Q.1. Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point *P* (5, 4, 2) to the line $\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k}).$ **Also find the image of *P* in this line.**

Ans.

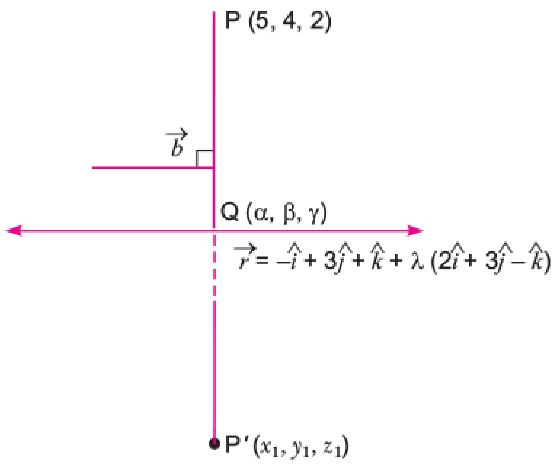
Given line is

$$\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$$

It can be written in cartesian form as

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} \quad ...(i)$$

Let Q (α, β, γ) be the foot of perpendicular drawn from $P(5, 4, 2)$ to the line (i) and P' (x_1, y_1, z_1) be the image of P on the line (i)



$\because$ $Q(\alpha, \beta, \gamma)$ lie on line (i)

$\therefore$ $\frac{\alpha+1}{2} = \frac{\beta-3}{3} = \frac{\gamma-1}{-1} = \lambda$ (say)

$\Rightarrow \alpha = 2\lambda - 1; \beta = 3\lambda + 3$ and $\gamma = -\lambda + 1$ ...(ii)

Now, $\overrightarrow{PQ} = (\alpha - 5)\hat{i} + (\beta - 4)\hat{j} + (\gamma - 2)\hat{k}$

Parallel vector of line (i) $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$.

Obviously $\overrightarrow{PQ} \perp \vec{b} \Rightarrow \overrightarrow{PQ} \cdot \vec{b} = 0$

$2(\alpha - 5) + 3(\beta - 4) + (-1)(\gamma - 2) = 0$

$\Rightarrow 2\alpha - 10 + 3\beta - 12 - \gamma + 2 = 0$

$\Rightarrow 2\alpha + 3\beta - \gamma - 20 = 0$

$\Rightarrow 2(2\lambda - 1) + 3(3\lambda + 3) - (-\lambda + 1) - 20 = 0$ [Putting value of α, β, γ from (ii)]

$\Rightarrow 4\lambda - 2 + 9\lambda + 9 + \lambda - 1 - 20 = 0$

$\Rightarrow 14\lambda - 14 = 0 \Rightarrow \lambda = 1$

Hence the coordinates of foot of perpendicular Q are $(2 \times 1 - 1,\ 3 \times 1 + 3, -1 + 1)$, *i.e.*, $(1, 6, 0)$

$\therefore$ Length of perpendicular $= \sqrt{(5-1)^2 + (4-6)^2 + (2-0)^2}$

$= \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$ units.

Also, since Q is mid-point of PP'

$\therefore 1 = \frac{x_1+5}{2} \Rightarrow x_1 = -3$

$6 = \frac{y_1+4}{2} \Rightarrow y_1 = 8$

$0 = \frac{z_1+2}{2} \Rightarrow z_1 = -2$

Therefore required image is $(-3, 8, -2)$.

Q.2. Find the equation of the plane that contains the point (1, – 1, 2) and is perpendicular to both the planes 2*x* + 3*y* – 2*z* = 5 and *x* + 2*y* – 3*z* = 8. Hence find the distance of point *P*(–2, 5, 5) from the plane obtained above.

Ans.

Equation of plane containing the point $(1, -1, 2)$ is given by

$a(x-1) + b(y+1) + c(z-2) = 0 \ldots(i)$

$\because$ (i) is perpendicular to plane $2x + 3y - 2z = 5$

$\therefore$ $2a + 3b - 2c = 0 \ldots(ii)$

Also, (i) is perpendicular to plane $x + 2y - 3z = 8$

$a + 2b - 3c = 0 \ldots(iii)$

From (ii) and (iii), we get

$\frac{a}{-9+4} = \frac{b}{-2+6} = \frac{c}{4-3}$

$\Rightarrow \frac{a}{-5} = \frac{b}{4} = \frac{c}{1} = \lambda$ (say)

$\Rightarrow$ $a = -5\lambda$, $b = 4\lambda$, $c = 1$

Putting these values in (i), we get

$-5\lambda(x-1) + 4\lambda(y+1) + \lambda(z-2) = 0$

$\Rightarrow -5(x-1) + 4(y+1) + (z-2) = 0$

$\Rightarrow -5x + 5 + 4y + 4 + z - 2 = 0 \Rightarrow -5x + 4y + z + 7 = 0$

$\Rightarrow 5x - 4y - z - 7 = 0 \ldots (iv)$ is the required equation of plane.

Again, if d be the distance of point $P(-2, 5, 5)$ to plane (iv), then

$$d = \left|\frac{5\times(-2)+(-4)\times5+(-1)\times5-7}{\sqrt{5^2+(-4)^2+(-1)^2}}\right|$$

$$= \left|\frac{-10-20-5-7}{\sqrt{25+16+1}}\right| = \frac{42}{\sqrt{42}} = \sqrt{42}\text{units}$$

Q.3. If the lines $\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2}$ and $\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$ are perpendicular, then find the value of *k* and hence find the equation of plane containing these lines.

Ans.

Given lines are

$$\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2} \quad ...(i)$$

$$\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5} \quad ...(ii)$$

Obviously, parallel vectors $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ of line (i) and (ii) respectively are:

$$\overrightarrow{b_1} = -3\hat{i} - 2k\hat{j} + 2\hat{k} \text{ and } \overrightarrow{b_2} = k\hat{i} + \hat{j} + 5\hat{k}$$

Lines $(i) \perp (ii) \Rightarrow \overrightarrow{b_1} \perp \overrightarrow{b_2}$

$$\Rightarrow \overrightarrow{b_1}.\overrightarrow{b_2} = 0 \quad \Rightarrow -3k - 2k + 10 = 0$$

$$\Rightarrow -5k + 10 = 0 \Rightarrow k = \frac{-10}{-5} = 2$$

Putting $k = 2$ in (i) and (ii), we get

$$\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2} \text{ and } \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{5}$$

Now, the equation of plane containing above two lines is

$$\begin{bmatrix} x-1 & y-2 & z-3 \\ -3 & -4 & 2 \\ 2 & 1 & 5 \end{bmatrix} = 0$$

$\Rightarrow (x-1)(-20-2) - (y-2)(-15-4) + (z-3)(-3+8) = 0$

$\Rightarrow -22(x-1) + 19(y-2) + 5(z-3) = 0$

$\Rightarrow -22x + 22 + 19y - 38 + 5z - 15 = 0$

$\Rightarrow -22x + 19y + 5z - 31 = 0 \Rightarrow \quad 22x - 19y - 5z + 31 = 0$

Note: Equation of plane containing lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and

$$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \text{ is } \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0, \text{ or } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Q.4. Find the vector equation of the plane passing through the points (2, 1, –1) and (–1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$. Also show that the plane thus obtained contains the line $\vec{r} = -\hat{i} + 3\hat{j} + 4\hat{k} + \lambda(3\hat{i} - 2\hat{j} - 5\hat{k})$.

Ans.

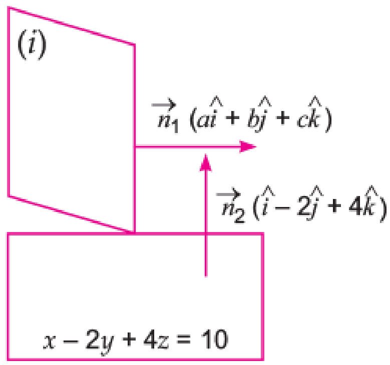
Let the equation of plane through $(2, 1, -1)$ be

$a(x-2) + b(y-1) + c(z+1) = 0 \ ...(i)$

$\because$ (i) passes through $(-1, 3, 4)$

$\therefore$ $a(-1-2) + b(3-1) + c(4+1) = 0$

$\Rightarrow \quad -3a + 2b + 5c = 0 \ ...(ii)$



Also plane (i) is perpendicular to plane $x - 2y + 4z = 10$

$\Rightarrow \overrightarrow{n_1} \perp \overrightarrow{n_2} \;\Rightarrow\; \overrightarrow{n_1} . \overrightarrow{n_2} = 0$

$\therefore\ 1\,a - 2b + 4c = 0$ ...(iii)

From (ii) and (iii), we get

$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2} \qquad \Rightarrow \qquad \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda$ (say)

$\Rightarrow a = 18\lambda,\ b = 17\lambda,\ c = 4\lambda$

Putting the value of a, b, c in (i), we get

$18\lambda\,(x - 2) + 17\lambda\,(y - 1) + 4\lambda\,(z + 1) = 0$

$\Rightarrow 18x - 36 + 17y - 17 + 4z + 4 = 0$

$\Rightarrow 18x + 17y + 4z = 49$

$\therefore$ Required vector equation of plane is

$\vec{r} . (18\hat{i} + 17\hat{j} + 4\hat{k}) = 49$ ...(iv)

Obviously plane (iv) contains the line

$\vec{r} = (-\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda\,(3\hat{i} - 2\hat{j} - 5\hat{k})$ ...(v)

Since, point $(-\hat{i}+3\hat{j}+4\hat{k})$ satisfy equation (iv) and vector $(18\hat{i}+17\hat{j}+4\hat{k})$ is perpendicular to, $(3\hat{i}-2\hat{j}+5\hat{k})$, as $(-\hat{i}+3\hat{j}+4\hat{k}).\,(18\hat{i}+17\hat{j}+4\hat{k}) = -18+51+16 = 49$

and $(18\hat{i}+17\hat{j}+4\hat{k}).\,(3\hat{i}-2\hat{j}-5\hat{k}) = 54-34-20 = 0$

Therefore, (iv) contains line (v).

Q.5. Let *P* (3, 2, 6) be a point in the space and *Q* be a point on the line $\vec{r} = (\hat{i}-\hat{j}+2\hat{k})+\mu(-3\hat{i}+\hat{j}+5\hat{k})$, then find the value of for which the vector μ is $\overrightarrow{PQ}$ parallel to the plane *x* – 4*y* + 3*z* = 1

Ans.

Let $P(3, 2, 6)$ be a point in the space and $Q(\alpha, \beta, \gamma)$ be a point on the given line represented in cartesian form as

$\frac{x-1}{-3} = \frac{y+1}{1} = \frac{z-2}{5} = \mu \quad \ldots(i)$

$Q(\alpha, \beta, \gamma)$ lie on line (i)

$\frac{\alpha-1}{-3} = \frac{\beta+1}{1} = \frac{\gamma-2}{5} = \mu$

$\alpha = -3\mu+1, \beta = \mu-1, \gamma = 5\mu+2 \ldots(ii)$

Now, $\overrightarrow{PQ} = (\alpha-3)\hat{i}+(\beta-2)\hat{j}+(\gamma-6)\hat{k}$

Normal vector of plane, $\vec{n} = \hat{i}-4\hat{j}+3\hat{k}$

Obviously, $\overrightarrow{PQ}$ is perpendicular to $\vec{n}$.

$\therefore \overrightarrow{PQ}\ \vec{n} = 0$

$(\alpha - 3).1 + (\beta - 2).(-4) + (\gamma - 6).3 = 0$

$\Rightarrow \alpha - 3 - 4\beta + 8 + 3\gamma - 18 = 0 \qquad \Rightarrow \qquad \alpha - 4\beta + 3\gamma - 13 = 0$

Putting the value of α, β, γ from (ii), we get

$-3\mu + 1 - 4(\mu - 1) + 3(5\mu + 2) - 13 = 0$

$\Rightarrow -3\mu + 1 - 4\mu + 4 + 15\mu + 6 - 13 = 0$

$\Rightarrow 8\mu - 2 = 0 \quad \Rightarrow \quad \mu = \frac{2}{8} = \frac{1}{4}$

Q.6. Find the vector and cartesian equations of the plane which bisects the line joining the points (3, –2, 1) and (1, 4, –3) at right angles.

Ans.

Let $P(3, -2, 1)$; $Q(1, 4, -3)$ be two points such that R (a point of plane) is mid point of $\overrightarrow{PQ}$ and $\overrightarrow{PQ}$ is perpendicular to required plane.

Now, coordinate of $R = \left(\frac{3+1}{2}, \frac{4-2}{2}, \frac{-3+1}{2}\right) = (2, 1, -1)$

Also, $\overrightarrow{PQ} = (1 - 3)\hat{i} + (4 + 2)\hat{j} + (-3 - 1)\hat{k} = -2\hat{i} + 6\hat{j} - 4\hat{k}$

Now, we have a normal vector $\overrightarrow{PQ}$ and a point $R(2, 1, -1)$ of required plane.

Therefore, vector equation of required plane is

$$\left(\vec{r} - (2\hat{i} + \hat{j} - \hat{k})\right).(-2\hat{i} + 6\hat{j} - 4\hat{k}) = 0$$

$$\{\vec{r} - (2\hat{i} + \hat{j} - \hat{k})\}.(\hat{i} - 3\hat{j} + 2\hat{k}) = 0$$

$$\Rightarrow \quad \vec{r}.(\hat{i} - 3\hat{j} + 2\hat{k}) - (2 - 3 - 2) = 0$$

$$\Rightarrow \quad \vec{r}.(\hat{i} - 3\hat{j} + 2\hat{k}) + 3 = 0$$

$$x - 3y + 2z + 3 = 0$$

Also, cartesian equation of required plane is

Q.7. Find the vector and Cartesian equations of a plane containing the two lines.

$$\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 5\hat{k}) \text{ and } \vec{r} = (3\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(3\hat{i} - 2\hat{j} + 5\hat{k})$$

Ans.

Given lines are

$$\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 5\hat{k}) \quad ...(i)$$

$$\vec{r} = (3\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(3\hat{i} - 2\hat{j} + 5\hat{k}) \quad ...(ii)$$

Here $\vec{a_1} = 2\hat{i} + \hat{j} + 3\hat{k}; \qquad \vec{a_2} = 3\hat{i} + 3\hat{j} + 2\hat{k}$

$\vec{b_1} = \hat{i} + 2\hat{j} + 5\hat{k}; \qquad \vec{b_2} = 3\hat{i} - 2\hat{j} + 5\hat{k}$

Now, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 5 \\ 3 & -2 & 5 \end{vmatrix}$

$= (10+10)\hat{i} - (5 - 15)\hat{j} + (-2 - 6)\hat{k} = 20\hat{i} + 10\hat{j} - 8\hat{k}$

Hence, vector equation of required plane is

$(\vec{r} - \vec{a_1}).(\vec{b_1} \times \vec{b_2}) = 0$

$\Rightarrow \vec{r}.(\vec{b_1} \times \vec{b_2}) = \vec{a_1}.(\vec{b_1} \times \vec{b_2})$

$\Rightarrow \vec{r}.(20\hat{i} + 10\hat{j} - 8\hat{k}) = (2\hat{i} + \hat{j} - 3\hat{k}).(20\hat{i} + 10\hat{j} - 8\hat{k})$

$\Rightarrow \vec{r}.(20\hat{i} + 10\hat{j} - 8\hat{k}) = 40 + 10 + 24$

$\vec{r}.(20\hat{i} + 10\hat{j} - 8\hat{k}) = 74 \quad \Rightarrow \quad \vec{r}.(20\hat{i} + 10\hat{j} - 8\hat{k}) = 74 \quad \Rightarrow \quad \vec{r}.(10\hat{i} + 5\hat{j} - 4\hat{k}) = 37$

Therefore, Cartesian equation is $10x + 5y - 4z = 37$

Q.8. Find the distance of the point (–2, 3, –4) from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

Ans.

Let given point be $P(-2, 3, -4)$ and given line and plane be

$\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5} \quad ...(i)$

$4x + 12y - 3z + 1 = 0 \quad ...(ii)$

Let $Q(\alpha, \beta, \gamma)$ be the point on line (i), such that

$\overrightarrow{PQ}$ parallel to plane (ii)

$\Rightarrow \overrightarrow{PQ} \perp \vec{n}$ [normal vector of (ii)]

Now, $\overrightarrow{PQ} = (\alpha + 2)\hat{i} + (\beta - 3)\hat{j} + (\gamma + 4)\hat{k}$

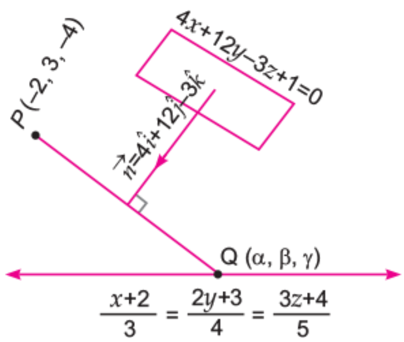
and $\vec{n} = 4\hat{i} + 12\hat{j} - 3\hat{k}$

$\therefore\ \overrightarrow{PQ}\,.\,\vec{n} = 0$

$\Rightarrow 4\,(\alpha + 2) + 12\,(\beta - 3) - 3(\gamma + 4) = 0$

$\Rightarrow 4\alpha + 8 + 12\beta - 36 - 3\gamma - 12 = 0$

$\Rightarrow 4\alpha + 12\beta - 3\gamma = 40 \qquad ...(iii)$



Also, $Q\,(\alpha, \beta, \gamma)$ lie on line (i)

$\frac{\alpha+2}{3} = \frac{2\beta+3}{4} = \frac{3\gamma+4}{5} = \lambda$ (say)

$\alpha = 3\lambda - 2,\ \beta = \frac{4\lambda - 3}{2},\ \gamma = \frac{5\lambda - 4}{3}$

Putting the value of α, β, γ in (iii), we get

$4\,(3\lambda - 2) + 12\left(\frac{4\lambda - 3}{2}\right) - 3\left(\frac{5\lambda - 4}{3}\right) = 40$

$\Rightarrow\ 12\lambda - 8 + 24\lambda - 18 - 15\lambda + 4 = 40$

$\Rightarrow\ 31\lambda - 22 = 40$

$\Rightarrow\ 31\lambda = 62$

$\Rightarrow \lambda = 2$

$\therefore\ \alpha = 3 \times 2 - 2 = 4,\ \beta = \frac{4\times 2 - 3}{2} = \frac{5}{2}, \quad \gamma = \frac{5\times 2 - 4}{3} = 2$

Hence $Q \equiv (4, \frac{5}{2}, 2)$

$\therefore$ Distance, $PQ = \sqrt{(4+2)^2 + \left(\frac{5}{2} - 3\right)^2 + (2+4)^2} = \sqrt{36 + 36 + \frac{1}{4}} = \sqrt{\frac{289}{4}} = \frac{17}{2}$ units.

Q.9. Find the equation of the plane passing through the point (–1, 3, 2) and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

Ans.

The equation of plane through $(-1, 3, 2)$ can be expressed as

$$A(x+1) + B(y-3) + C(z-2) = 0$$

As the required plane is perpendicular to $x + 2y + 3z = 5$ and $3x + 3y + z = 0$, we get

$$A + 2B + 3C = 0 \quad \text{and } 3A + 3B + C = 0$$

$$\Rightarrow \frac{A}{2-9} = \frac{B}{9-1} = \frac{C}{3-6} \qquad \Rightarrow \qquad \frac{A}{-7} = \frac{B}{8} = \frac{C}{-3}$$

$\therefore$ Direction ratios of normal to the required plane are $-7, 8, -3$.

Hence, equation of the plane will be

$$-7(x+1) + 8(y-3) - 3(z-2) = 0$$

$$\Rightarrow \qquad -7x - 7 + 8y - 24 - 3z + 6 = 0$$

or $\qquad 7x - 8y + 3z + 25 = 0$

Q.10. Find the distance of the point (2, 12, 5) from the point of intersection of the line

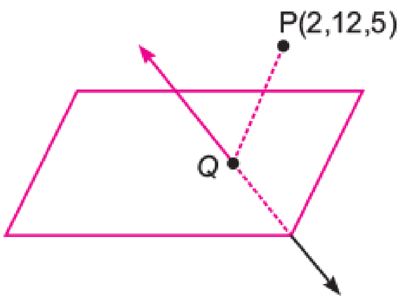
$\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ **and the plane** $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0.$

Ans.

Given line and plane are

$$\vec{r} = (2\hat{i} - 4\hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \quad \ldots (i)$$

and $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0 \quad \ldots (ii)$



For intersection point Q, we solve equations (i) and (ii) by putting the value of $\vec{r}$ from (i) in (ii)

$$[(2\hat{i} - 4\hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})].\ (\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow\ [(2 + 3\lambda)\hat{i} - (4 - 4\lambda)\hat{j} + (2 + 2\lambda)\hat{k}].(\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow (2+3\lambda) + 2\ (4 - 4\lambda) + (2 + 2\lambda) = 0$$

$$\Rightarrow 2 + 3\lambda + 8 - 8\lambda + 2 + 2\lambda = 0 \Rightarrow 12 - 3\lambda = 0$$

$$\Rightarrow \lambda = 4$$

Hence position vector of intersecting point is $14\hat{i} + 12\hat{i} + 10\hat{k}$.

Co-ordinate of intersecting point, $Q \equiv (14, 12, 10)$

Required distance = $\sqrt{(14 - 2)^2 + (12 - 12)^2 + (10 - 5)^2}$

$= \sqrt{144 + 25} = \sqrt{169}$ units

$= 13$ units.

Q.11. The points *A* (4, 5, 10), *B* (2, 3, 4) and C (1, 2, –1) are three vertices of a parallelogram *ABCD*. Find the vector equations of the sides *AB* and *BC* and also find the coordinates of point *D*.

Ans.

The points $A\ (4, 5, 10)$, $B\ (2, 3, 4)$ and $C\ (1, 2, -1)$ are three vertices of parallelogram $ABCD$.

Let coordinates of D be (x, y, z)

Direction vector along AB is

$$\vec{a} = (2-4)\hat{i} + (3-5)\hat{j} + (4-10)\hat{k} = -2\hat{i} - 2\hat{j} - 6\hat{k}$$

$\therefore$ Equation of line AB, is given by

$$\vec{b} = (4\hat{i} + 5\hat{j} + 10\hat{k}) + \lambda(2\hat{i} + 2\hat{j} + 6\hat{k})$$

Direction vector along BC is

$$\vec{c} = (1-2)\hat{i} + (2-3)\hat{j} + (-1-4)\hat{k} = -\hat{i} - \hat{j} - 5\hat{k}$$

$\therefore$ Equation of a line BC, is given by .

$$\vec{d} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \mu(\hat{i} + \hat{j} + 5\hat{k})$$

Since $ABCD$ is a parallelogram AC and BD bisect each other

$$\therefore \left[\frac{4+1}{2}, \frac{5+2}{2}, \frac{10-1}{2}\right] = \left[\frac{2+x}{2}, \frac{3+y}{2}, \frac{4+z}{2}\right]$$

$\Rightarrow 2 + x = 5,\ 3 + y = 7,\ 4 + z = 9$

$\Rightarrow x = 3,\ y = 4,\ z = 5$

Coordinates of D are $(3, 4, 5)$.

Q.12. Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point *P* (3, 2, 1) from the plane $2x - y + z + 1 = 0$. Also find the image of the point in the plane.

Ans.

Let $O(\alpha, \beta, \gamma)$ be the image of the point $P(3, 2, 1)$ in the plane

$2x - y + z + 1 = 0$

PO is perpendicular to the plane and S is the mid-point of PO and the foot of the perpendicular.

Dr's of PS are 2, −1, 1.

∴ Equation of PS are $\frac{x-3}{2} = \frac{y-2}{-1} = \frac{z-1}{1} = \mu$

∴ General point on line is $S(2\mu + 3, -\mu + 2, \mu + 1)$

If this point lies on plane, then

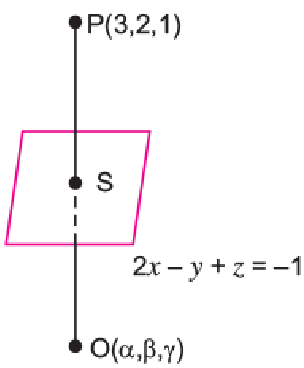
$2(2\mu + 3) - (-\mu + 2) + 1(\mu + 1) + 1 = 0$

$\Rightarrow 6\mu + 6 = 0 \quad \Rightarrow \quad \mu = -1$

∴ Coordinates of S are $(1, 3, 0)$.

As S is the mid point of PO,

∴The coordinate of $S = \left(\frac{3+\alpha}{2}, \frac{2+\beta}{2}, \frac{1+\gamma}{2}\right) = (1, 3, 0)$



By comparing both sides, we get

$\frac{3+\alpha}{2} = 1 \quad \Rightarrow \quad \alpha = -1$

$\frac{2+\beta}{2} = 3 \quad \Rightarrow \quad \beta = 4$

$\frac{1+\gamma}{2} = 0 \quad \Rightarrow \quad \gamma = -1$

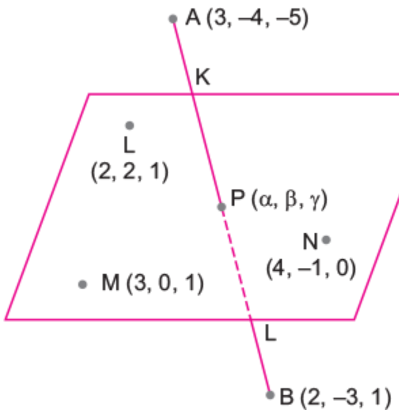
Image of point P is $(-1, 4, -1)$.

Q.13. Find the coordinate of the point *P* where the line through *A*(3, – 4, – 5) and *B* (2, – 3, 1) crosses the plane passing through three points *L*(2, 2, 1), *M*(3, 0, 1) and *N*(4, –1, 0). Also, find the ratio in which *P* divides the line segment AB.

Ans.

Let the coordinate of *P* be (α, β, γ).

Equation of plane passing through *L*(2, 2, 1), *M*(3, 0, 1) and *N*(4, –1, 0) is given by



$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 1-1 \\ 4-2 & -1-2 & 0-1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(2-0)-(y-2)(-1-0)+(z-1)(-3+4)=0$$

$$\Rightarrow 2(x-2)+(y-2)+z-1=0$$

$$\Rightarrow 2x-4+y-2+z-1=0$$

$$\Rightarrow 2x+y+z-7=0 \ ...(i)$$

Now, the equation of line passing through $A(3, -4, -5)$ and $B(2, -3, 1)$ is given by

$\Rightarrow \frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} \quad ...(ii)$

$\because \ P(\alpha, \beta, \gamma)$ lie on line AB

$\Rightarrow \frac{\alpha-3}{-1} = \frac{\beta+4}{1} = \frac{\gamma+5}{6} = \lambda$ (say)

$\Rightarrow$ a = $-\lambda + 3$, b = $\lambda - 4$, g = $6\lambda - 5$

Also$P(\alpha, \beta, \gamma)$ lie on plane (i)

$\Rightarrow 2\alpha + \beta + \gamma - 7 = 0$

$\Rightarrow 2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) - 7 = 0$

$\Rightarrow -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 - 7 = 0$

$\Rightarrow 5\lambda - 10 = 0$

$\Rightarrow \lambda = 2$

$\therefore \alpha = 1, \beta = -2, \gamma = 7$

$\therefore$ Co-ordinate of $P \equiv (1, -2, 7)$

Let P divides AB in the ratio $K : 1$.

$\therefore \ 1 = \frac{K \times 2 + 1 \times 3}{K+1}$

$\Rightarrow K + 1 = 2K + 3 \Rightarrow K = -2$

$\Rightarrow$ P divides AB externally in the ratio 2 : 1.

Q.14. Find the shortest distance between the lines $x + 1 = 2y = -12z$ and $x = y + 2 = 6z - 6$.

Ans.

Given lines are

$x+1=2y=-12z$ and $x = y + 2 = 6z-6$

$\Rightarrow \quad \frac{x-(-1)}{1}=\frac{y-0}{\frac{1}{2}}=\frac{z-0}{-\frac{1}{12}}$ and $\frac{x-0}{1}=\frac{y-(-2)}{1}=\frac{z-1}{\frac{1}{6}}$

These lines may be written in vector form as

$\vec{r}=(-\hat{i}+0\hat{j}+0\hat{k})+\lambda(\hat{i}+\frac{1}{2}\hat{j}-\frac{1}{12}\hat{k}) \quad ...(i)$

and $\vec{r}=(0\hat{i}-2\hat{j}+\hat{k})+\lambda(\hat{i}+\hat{j}+\frac{1}{6}\hat{k}) ...(ii)$

We know that the shortest distance between

$\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$ is given by

$$\text{SD}=\left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$$

Here, $\vec{a_1}=-\hat{i}+0\hat{j}+0\hat{k}, \qquad \vec{b_1}=\hat{i}+\frac{1}{2}\hat{j}-\frac{1}{12}\hat{k}$

$\vec{a_2}=0\hat{i}-2\hat{j}+\hat{k}, \qquad \vec{b_2}=\hat{i}+\hat{j}+\frac{1}{6}\hat{k}$

Now, $\vec{a_2}-\vec{a_1}=(0\hat{i}-2\hat{j}+\hat{k})-(-\hat{i}+0\hat{j}+0\hat{k})=\hat{i}-2\hat{j}+\hat{k}$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1/2 & -1/12\\ 1 & 1 & 1/6\end{vmatrix}=\left(\frac{1}{12}+\frac{1}{12}\right)\hat{i}-\left(\frac{1}{6}+\frac{1}{12}\right)\hat{j}+\left(1-\frac{1}{2}\right)\hat{k}$$

$$=\frac{1}{6}\hat{i}-\frac{1}{4}\hat{j}+\frac{1}{2}\hat{k}$$

$$\therefore \quad |\vec{b_1}\times\vec{b_2}|=\sqrt{\left(\frac{1}{6}\right)^2+\left(-\frac{1}{4}\right)^2+\left(\frac{1}{2}\right)^2}$$

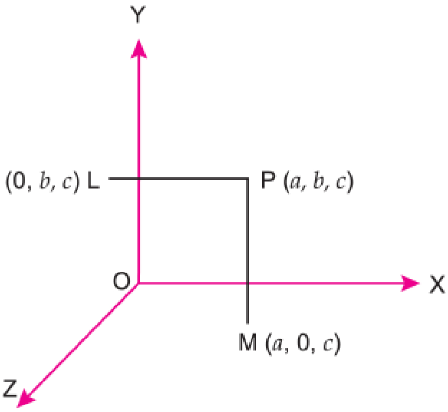
$$=\sqrt{\frac{1}{36}+\frac{1}{16}+\frac{1}{4}}=\sqrt{\frac{4+9+36}{144}}=\sqrt{\frac{49}{144}}=\frac{7}{12}$$

$$\therefore \text{ Required } S.D.=\left|\frac{(\hat{i}-2\hat{j}+\hat{k}).\left(\frac{1}{6}\hat{i}-\frac{1}{4}\hat{j}+\frac{1}{2}\hat{k}\right)}{\frac{7}{12}}\right|=\left|\frac{\frac{1}{6}+\frac{1}{2}+\frac{1}{2}}{\frac{7}{12}}\right|=\frac{\frac{7}{6}}{\frac{7}{12}}=\frac{7}{6}\times\frac{12}{7}=2 \text{ units}.$$

Q.15. From the point *P*(*a*, *b*, *c*), perpendiculars *PL* and *PM* are drawn to *YZ* and *ZX* planes respectively. Find the equation of the plane *OLM*.

Ans.

Obviously, the coordinates of *O*, *L* and *M* are (0, 0, 0), (0, *b*, *c*) and (*a*, 0, *c*).



Therefore, the equation of required plane is given by

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 0-0 & b-0 & c-0 \\ a-0 & 0-0 & c-0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x & y & z \\ 0 & b & c \\ a & 0 & c \end{vmatrix} = 0$$

$$\Rightarrow \; x(bc-0) - y(0-ac) + z(0-ab) = 0$$

$$\Rightarrow \quad bcx + acy - abz = 0$$

Q.16. Find the equation of the plane passing through the point (1, 1, –1) and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

Ans.

Let the equation of plane passing through point $(1, 1, -1)$ be

$a(x-1) + b(y-1) + c(z+1) = 0 \quad ...(i)$

Since (i) is perpendicular to the plane $x + 2y + 3z - 7 = 0$

$\therefore \quad 1.a + 2.b + 3.c = 0 \quad \Rightarrow a + 2b + 3c = 0 \quad ...(ii)$

Again plane (i) is perpendicular to the plane $2x - 3y + 4z = 0$

$\therefore \quad 2.a - 3.b + 4.c = 0 \quad \Rightarrow 2a - 3b + 4c = 0 \quad ...(iii)$

From (ii) and (iii), we get

$$\frac{a}{8+9} = \frac{b}{6-4} = \frac{c}{-3-4} \quad \Rightarrow \quad \frac{a}{17} = \frac{b}{2} = \frac{c}{-7} = \lambda$$

$\Rightarrow a = 17\lambda, b = 2\lambda, c = -7\lambda$

Putting the value of a, b, c in (i), we get

$17\lambda(x-1) + 2\lambda(y-1) - 7\lambda(z+1) = 0$

$\Rightarrow 17(x-1) + 2(y-1) - 7(z+1) = 0$

$\Rightarrow 17x + 2y - 7z - 17 - 2 - 7 = 0$

$\Rightarrow 17x + 2y - 7z - 26 = 0$ is the required equation.

[**Note:** The equation of plane passing through (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b, c are direction ratios of normal of plane.]

Q.17. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Also find the distance of the plane obtained above, from the origin.

Ans.

The equation of a plane passing through the intersection of the given planes is

$(x + y + z - 1) + \lambda\ (2x + 3y + 4z - 5) = 0$

$\Rightarrow (1 + 2\lambda)\ x + (1 + 3\lambda)\ y + (1 + 4\lambda)\ z - (1+5\lambda) = 0\ \ldots (i)$

Since, (i) is perpendicular to $x - y + z = 0$

$\Rightarrow (1 + 2\lambda)\ 1 + (1 + 3\lambda)\ (-1) + (1 + 4\lambda)\ 1 = 0$

$\Rightarrow 1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$

$\Rightarrow\ 3\lambda + 1 = 0$

$\Rightarrow\ \lambda = -\frac{1}{3}$

Putting the value of l in (i), we get

$(1-\frac{2}{3})x + (1-1)y + (1-\frac{4}{3})z - (1-\frac{5}{3}) = 0 \quad \Rightarrow \qquad \frac{x}{3} - \frac{z}{3} + \frac{2}{3} = 0$

$\Rightarrow x - z + 2 = 0$, it is required plane.

Let d be the distance of this plane from origin.

$\therefore\ d = \left|\frac{0.\ x\ +\ 0.\ y + 0.\ (-z) + 2}{\sqrt{1^2 + 0^2 + (-1)^2}}\right| = \left|\frac{2}{\sqrt{2}}\right| = \sqrt{2}$ units.

[**Note:** The distance of the point (α, β, γ) to the plane $ax + by + cz + d = 0$ is given by $\left|\frac{a\alpha + b\beta + c\gamma + d}{\sqrt{a^2 + b^2 + c^2}}\right|$.

Q.18. Find the coordinates of the point where the line through the points *A* (3, 4, 1) and *B* (5, 1, 6) crosses the *XY*-plane.

Ans.

Let P (α, β, γ) be the point at which the given line crosses the XY plane

Now, the equation of given line is

$$\frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5} \quad \ldots(i)$$

Since P (α, β, γ) lies on line (i)

$$\therefore \quad \frac{\alpha-3}{2}=\frac{\beta-4}{-3}=\frac{\gamma-1}{5}=\lambda \text{ (say)}$$

$\Rightarrow$ $\alpha = 2\lambda + 3$; $\beta = -3\lambda + 4$ and $\gamma = 5\lambda + 1$

Also P (α, β, γ) lie on given XY plane, *i.e.*, $z = 0$

$\therefore$ $0.\alpha + 0.\beta + \gamma = 0$

$\Rightarrow$ $5\lambda + 1 = 0$

$\Rightarrow$ $\lambda = -\frac{1}{5}$.

Hence, the coordinates of required point is

$$\alpha = 2\times\left(-\frac{1}{5}\right)+3=\frac{13}{5};\ \beta = -3\times\left(-\frac{1}{5}\right)+4=\frac{23}{5} \quad \text{and} \quad \gamma=5\times\left(-\frac{1}{5}\right)+1=0$$

i.e., required coordinates are $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$.

Q.19. Find the vector equation of the plane passing through the points (3, 4, 2) and (7, 0, 6) and perpendicular to the plane 2*x* – 5*y* – 15 = 0. Also show that the plane thus obtained contains the line $\vec{r}=\hat{i}+3\hat{j}-2\hat{k}+\lambda(\hat{i}-\hat{j}+\hat{k})$.

Ans.

Let the equation of plane through $(3, 4, 2)$ be

$a(x-3) + b(y-4) + c(z-2) = 0 \quad ...(i)$

$\because$ (i) passes through $(7, 0, 6)$

$\therefore$ $a(7-3) + b(0-4) + c(6-2) = 0 \Rightarrow 4a - 4b + 4c = 0$

$\Rightarrow a - b + c = 0 \; ...(ii)$

Also, since plane (i) is perpendicular to plane $2x - 5y - 15 = 0$

$2a - 5b + 0c = 0 \; ...(iii)$

From (ii) and (iii), we get

$\frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = \lambda$ (say) $\Rightarrow a = 5\lambda, b = 2\lambda, c = -3\lambda.$

Putting the value of a, b, c in (i), we get

$5\lambda(x-3) + 2\lambda(y-4) - 3\lambda(z-2) = 0$

$\Rightarrow 5x - 15 + 2y - 8 - 3z + 6 = 0$

$\Rightarrow 5x + 2y - 3z = 17$

$\therefore$ Required vector equation of plane is $\vec{r}.(5\hat{i} + 2\hat{j} - 3\hat{k}) = 17 \quad ...(iv)$

Obviously, plane (iv) contains the line

$\vec{r} = (\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \quad ...(v)$

Since, point $(\hat{i} + 3\hat{j} - 2\hat{k})$ satisfy the equation (iv) and vector is perpendicular to $(5\hat{i} + 2\hat{j} - 3\hat{k})$.

as $(\hat{i} + 3\hat{j} - 2\hat{k}).(5\hat{i} + 2\hat{j} - 3\hat{k}) = 5 + 6 + 6 = 17$ and $(5\hat{i} + 2\hat{j} - 3\hat{k}).(\hat{i} - \hat{j} + \hat{k}) = 5 - 2 - 3 = 0$

Therefore, (iv) contains line (v).

Q.20. Show that the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k});$$

$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k});$$

are intersecting. Hence find their point of intersection.

Ans.

Given lines are

$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = 5\hat{i} + 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$

Its corresponding cartesian forms are

$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z+4}{2} \quad ...(i)$

$\frac{x-5}{3} = \frac{y+2}{2} = \frac{z-0}{6} \quad ...(ii)$

If two lines (i) and (ii) intersect, let interesting point be .

$\Rightarrow$ (α, β, γ) satisfy line (i)

$\therefore \frac{\alpha-3}{1} = \frac{\beta-2}{2} = \frac{\gamma+4}{2} = \lambda = \text{(say)}$

$\Rightarrow \alpha = \lambda + 3, \beta = 2\lambda + 2, \gamma = 2\lambda - 4$

Also, (α, β, γ) will satisfy line (ii)

$\therefore \quad \frac{\alpha-5}{3} = \frac{\beta+2}{2} = \frac{\gamma}{6}$

$\Rightarrow \quad \frac{\lambda+3-5}{3} = \frac{2\lambda+2+2}{2} = \frac{2\lambda-4}{6}$

$\therefore \quad \frac{\lambda-2}{3} = \frac{\lambda+2}{1} = \frac{\lambda-2}{3}$

I II III

I and II $\Rightarrow \frac{\lambda-2}{3}=\frac{\lambda+2}{1} \Rightarrow \lambda-2=3\lambda+6 \Rightarrow \lambda=-4$

II and III $\Rightarrow \frac{\lambda+2}{1}=\frac{\lambda-2}{3} \Rightarrow \lambda=-4$

$\therefore$ The value of l is same in both cases.

Hence, both lines intersect each other at point

$(\alpha, \beta, \gamma) \equiv (-4+3, 2\times(-4)+2, 2(-4)-4)$ $(-1, -6, -12)$

Q.21. Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ **and** $\hat{i}+2\hat{j}+\hat{k}.$ **Also find the coordinates of the point of intersection of this plane and the line** $\vec{r}=3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k}).$

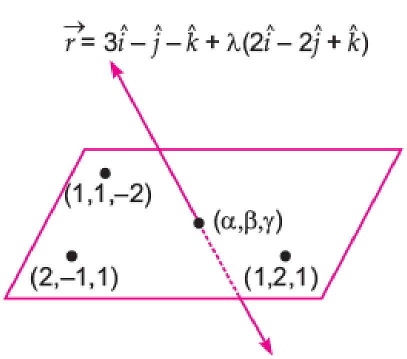
Ans.

The equation of plane passing through three points $\hat{i}+\hat{j}-2\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$

i.e., $(1, 1, -2)$, $(2, -1, 1)$ and $(1, 2, 1)$ is

$$\begin{vmatrix} x-1 & y-1 & z+2 \\ 2-1 & -1-1 & 1+2 \\ 1-1 & 2-1 & 1+2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z+2 \\ 1 & -2 & 3 \\ 0 & 1 & 3 \end{vmatrix}=0$$



$\Rightarrow\ (x-1)(-6-3)-(y-1)\ (3-0)+(z+2)\ (1+0)=0$

$\Rightarrow -9x+9-3y+3+z+2=0$

$\Rightarrow\ 9x+3y-z=14\ \ldots(i)$

Its vector form is $\overrightarrow{r}.\ (9\hat{j}+3\hat{j}-\hat{k})=14$

The given line is $\overrightarrow{r}=(3\hat{i}-\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$

Its cartesian form is

$\frac{x-3}{2}=\frac{y+1}{-2}=\frac{z+1}{1}\quad \ldots(ii)$

Let the line (ii) intersect plane (i) at (α, β, γ)

$\because\ (\alpha, \beta, \gamma)$ lie on (ii)

$\frac{\alpha-3}{2}=\frac{\beta+1}{-2}=\frac{\gamma+1}{1}=\lambda$ (say)

$\Rightarrow \alpha=2\lambda+3;\ \beta=-2\lambda-1;\ \gamma=\lambda-1$

Also, point (α, β, γ) lie on plane (i)

$\Rightarrow 9\,a+3b-g=14$

$\Rightarrow 9\,(2\lambda+3)+3\,(-2\lambda-1)-(\lambda-1)=14$

$\Rightarrow 18\lambda+27-6\lambda-3-\lambda+1=14$

$\Rightarrow 11\lambda+25=14$

$\Rightarrow 11\lambda=14-25$

$\Rightarrow 11\lambda=-11$

$\Rightarrow \lambda=-1$

Therefore, point of intersection $\equiv (1, 1, -2)$.

Q.22. Find the coordinates of the point where the line through the points *A* (3, 4, 1) and *B* (5, 1, 6) crosses the plane determined by the points *P* (2, 1, 2), *Q* (3, 1, 0) and *R* (4, –2, 1).

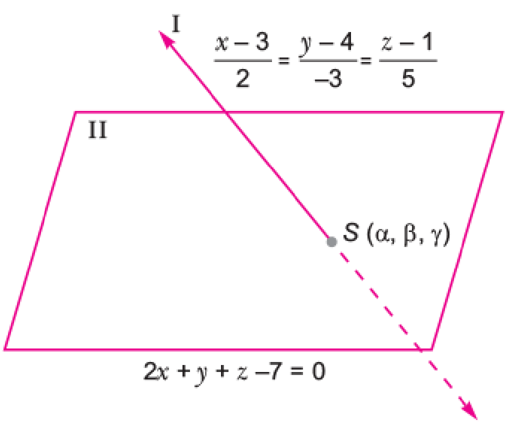
Ans.

The line through A (3, 4, 1) and B (5, 1, 6) is given by

$$\frac{x-3}{5-3} = \frac{y-4}{1-4} = \frac{z-1}{6-1} \quad \Rightarrow \quad \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5} \quad ...(i)$$

The equation of plane determined by the points P (2, 1, 2), Q (3, 1, 0) and R (4, −2, 1) is given by,

$$\begin{vmatrix} x-2 & y-1 & z-2 \\ 3-2 & 1-1 & 0-2 \\ 4-2 & -2-1 & 1-2 \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} x-2 & y-1 & z-2 \\ 1 & 0 & -2 \\ 2 & -3 & -1 \end{vmatrix} = 0$$



$$\Rightarrow (0-6)(x-2) - (-1+4)(y-1) + (-3-0)(z-2) = 0$$

$$\Rightarrow -6x + 12 - 3y + 3 - 3z + 6 = 0$$

$$\Rightarrow -6x - 3y - 3z + 21 = 0$$

$$\Rightarrow 2x + y + z - 7 = 0 \;...(ii)$$

Let $S(\alpha, \beta, \gamma)$ be intersecting point of line (I) and plane (II)

$\because$ $S(\alpha, \beta, \gamma)$ lie on line (I)

$\frac{\alpha - 3}{2} = \frac{\beta - 4}{-3} = \frac{\gamma - 1}{5} = \lambda$

$\therefore$ $\alpha = 2\lambda + 3, \beta = -3\lambda + 4, \gamma = 5\lambda + 1$

$\because$ $S(\alpha, \beta, \gamma)$ also lie on plane (II)

2a + b + g − 7 = 0

$\Rightarrow 2(2\lambda + 3) + (-3\lambda + 4) + (5\lambda + 1) - 7 = 0$

$\Rightarrow 4\lambda + 6 - 3\lambda + 4 + 5\lambda + 1 - 7 = 0$

$\Rightarrow 6\lambda + 4 = 0 \quad \Rightarrow \quad \lambda = -\frac{4}{6} = -\frac{2}{3}$

$\therefore \alpha = 2 \times \left(\frac{-2}{3}\right) + 3 = -\frac{4}{3} + 3 - \frac{5}{3}$

$\beta = -3 \times \left(-\frac{2}{3}\right) + 4 = 2 + 4 = 6$ and $\gamma = 5 \times \left(\frac{-2}{3}\right) + 1 = \frac{-10}{3} + 1 = -\frac{7}{3}$

$\therefore$ Required point of intersection = $\left(\frac{5}{3}, 6, -\frac{7}{3}\right)$.

Q.23. Find the direction ratios of the normal to the plane, which passes through the points (1, 0, 0) and (0, 1, 0) and makes angle $\frac{\pi}{4}$ with the plane $x + y = 3$. Also find the equation of the plane.

Ans.

Let the equation of plane passing through the point $(1, 0, 0)$ be

$a(x-1) + b(y-0) + c(z-0) = 0$

$\Rightarrow ax - a + by + cz = 0$

$\Rightarrow ax + by + cz = a \ldots (i)$

Since, (i) also passes through $(0, 1, 0)$

$\Rightarrow 0 + b + 0 = a$

$\Rightarrow b = a \ldots (ii)$

Given, the angle between plane (i) and plane $x + y = 3$ is $\frac{\pi}{4}$.

$\therefore \cos\frac{\pi}{4} = \left|\frac{a.1+b.1+c.0}{\sqrt{a^2+b^2+c^2}\sqrt{1^2+1^2}}\right|$

$\Rightarrow \frac{1}{\sqrt{2}} = \left|\frac{a+b}{\sqrt{a^2+b^2+c^2}\sqrt{1+1}}\right|$

$\Rightarrow \frac{1}{\sqrt{2}} = \left|\frac{a+b}{\sqrt{a^2+b^2+c^2}\sqrt{2}}\right| \Rightarrow 1 = \left|\frac{a+b}{\sqrt{a^2+b^2+c^2}}\right|$

$\Rightarrow \sqrt{a^2+b^2+c^2} = \pm(a+b) \Rightarrow a^2+b^2+c^2 = (a+b)^2$

$\Rightarrow a^2 + b^2 + c^2 = a^2 + b^2 + 2ab$

$\Rightarrow c^2 = 2ab$

$\Rightarrow c^2 = 2a^2$ [From (ii)]

$\Rightarrow \sqrt{a^2+b^2+c^2} = \pm(a+b) \Rightarrow a^2+b^2+c^2 = (a+b)^2$

Now, equation (i) becomes

$\mathrm{ax} + \mathrm{ay} \pm \sqrt{2}\mathrm{az} = a$

$\Rightarrow x + y \pm \sqrt{2}z = 1$, is the required equation of plane.

Therefore, required direction ratios are $1, 1, \pm\sqrt{2}$.

Q.24. Find the equation of the plane which contains the line of intersection of the planes

$$\vec{r}.\,(\hat{i}-2\hat{j}+3\hat{k})-4 = 0 \text{ and } \vec{r}.\,(-2\hat{i}+\hat{j}+\hat{k})+5=0$$

and whose intercept on *x*-axis is equal to that of on *y*-axis.

Ans.

Given planes are $\vec{r}.\,(\hat{i}-2\hat{j}+3\hat{k})-4=0$ and $\vec{r}.\,(-2\hat{i}+\hat{j}+\hat{k})+5=0$

These can be written in cartesian form as

$x-2y+3z-4=0 \quad ...(i)$

and $-2x+y+z+5=0 \quad ...(ii)$

Now the equation of plane containing the line of intersection of the planes (i) and (ii) is given by

$(x-2y+3z-4)+\lambda(-2x+y+z+5)=0 \quad ...(iii)$

$\Rightarrow (1-2\lambda)\,x-(2-\lambda)\,y+(3+\lambda)\,z-4+5\lambda=0$

$\Rightarrow (1-2\lambda)\,x-(2-\lambda)\,y+(3+\lambda)\,z=4-5\lambda$

$\Rightarrow \dfrac{x}{\frac{4-5\lambda}{1-2\lambda}}+\dfrac{y}{\frac{4-5\lambda}{-2+\lambda}}+\dfrac{z}{\frac{4-5\lambda}{3+\lambda}}=1$

According to question $\dfrac{4-5\lambda}{1-2\lambda}=\dfrac{4-5\lambda}{-2+\lambda}$

$\Rightarrow 1-2\lambda=-2+\lambda$

$\Rightarrow 3\lambda=3$

$\Rightarrow \lambda=1$

Putting the value of $\lambda = 1$ in (iii), we get

$(x - 2y + 3z - 4) + 1\ (-2x + y + z + 5) = 0$

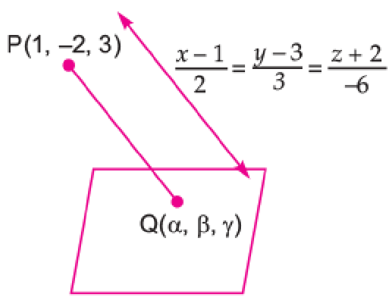
$-x - y + 4z + 1 = 0\ \vec{r}.(\hat{i} + \hat{j} - 4\hat{k}) - 1 = 0$

$\Rightarrow\ x + y - 4z - 1 = 0$

$\Rightarrow$ Its vector form is

Q.25. Find the distance of the point (1, –2, 3) from the plane *x – y + z* = 5 measured parallel to the line whose direction cosines are proportional to 2, 3, –6.

Ans.



Let $Q\ (\alpha, \beta, \gamma)$ be the point on the given plane

$x - y + z = 5 \qquad ...(i)$

Since PQ is parallel to given line

$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6} \qquad ...(ii)$ where $P\ (1, -2, 3)$ is the given point.

$\because\ PQ$ is parallel to given line (ii).

$\therefore\ \overrightarrow{PQ} \parallel \vec{b}$ (parallel vector of line).

$$\Rightarrow \frac{\alpha - 1}{2} = \frac{\beta + 2}{3} = \frac{\gamma - 3}{-6} = \lambda$$

$$\Rightarrow \alpha = 2\lambda + 1, \beta = 3\lambda - 2, \gamma = -6\lambda + 3$$

Now, $\because Q(\alpha, \beta, \gamma)$ lie on plane (i)

$$\alpha - \beta + \gamma = 5$$

$$2\lambda + 1 - 3\lambda + 2 - 6\lambda + 3 = 5$$

$$-7\lambda + 6 = 5 \qquad \Rightarrow -7\lambda = -1$$

$$\lambda = \frac{1}{7}$$

$$\alpha = 2 \times \frac{1}{7} + 1 = \frac{9}{7}; \beta = 3 \times \frac{1}{7} - 2 = -\frac{11}{7} \text{ and } \gamma = -6 \times \frac{1}{7} + 3 = \frac{15}{7}$$

Therefore required distance

$$PQ = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(-\frac{11}{7} + 2\right)^2 + \left(\frac{15}{7} - 3\right)^2} \text{ unit.}$$
$$= \sqrt{\frac{4}{49} + \frac{9}{49} \frac{36}{49}} = \sqrt{1} = 1$$

Q.26. Find the value of *p*, so that the lines $l_1 = \frac{1 - x}{3} = \frac{7y - 14}{p} = \frac{z - 3}{2}$ **and** $l_2 : \frac{7 - 7x}{3p} = \frac{y - 5}{1} = \frac{6 - z}{5}$ **are perpendicular to each other. Also find the equations of a line passing through a point (3, 2, –4) and parallel to line** l_1**.**

Ans.

Given line l_1 and l_2 are

$$l_1 \equiv \frac{1 - x}{3} = \frac{7y - 14}{p} = \frac{z - 3}{2}$$
$$\Rightarrow \quad \frac{x - 1}{-3} = \frac{y - 2}{\frac{p}{7}} = \frac{z - 3}{2}$$

$$l_2 \equiv \frac{7 - 7x}{3p} = \frac{y - 5}{1} = \frac{6 - z}{5}$$
$$\Rightarrow \quad \frac{x - 1}{\frac{-3p}{7}} = \frac{y - 5}{1} = \frac{z - 6}{-5}$$

Since $l_1 \perp l_2$

$$\Rightarrow (-3)\left(-\frac{3p}{7}\right) + \frac{p}{7} \times 1 + 2 \times (-5) = 0 \qquad \Rightarrow \frac{9p}{7} + \frac{p}{7} - 10 = 0 \quad \Rightarrow \quad \frac{10p}{7} = 10$$

$$\Rightarrow p = \frac{7 \times 10}{10} \qquad \Rightarrow \qquad p = 7$$

The equation of line passing through $(3, 2, -4)$ and parallel to l_1 is given by

$$\frac{x-3}{-3} = \frac{y-2}{\frac{p}{7}} = \frac{z+4}{2}$$

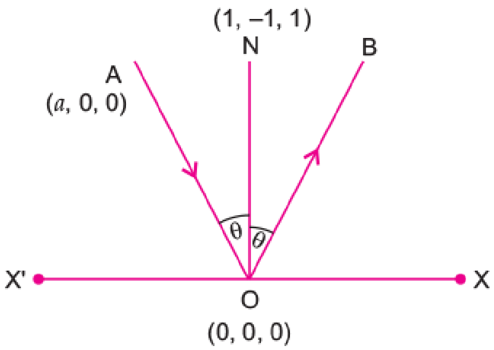
i.e., $\frac{x-3}{-3} = \frac{y-2}{1} = \frac{z+4}{2} \qquad (\because p = 7)$

Long Answer Questions-II (OIQ)

[6 Mark]

Q.1. A mirror and source of light are situated at the origin *O* and at a point on *OX* respectively. A ray of light from the source strike the mirror and is reflected. If the Dr's of the normal to the plane are 1, –1, 1, then find dc's of reflected ray.

Ans.



Let the source of light be situated at $(a, 0, 0)$ and AO and OB be incident and reflected rays. ON is the normal to the mirror at O.

Now Dr's at OA are $(a - 0)$, $(0 - 0)$, $(0 - 0)$ *i.e.,* $a, 0, 0$

$\therefore$ Dc's of OA $\frac{a}{\sqrt{a^2+0^2+0^2}}, \frac{0}{\sqrt{a^2+0^2+0^2}}, \frac{0}{\sqrt{a^2+0^2+0^2}}$ *i.e.,* 1, 0, 0.

Given, Dr's of ON are $1, -1, 1$

$\therefore$ Dc's of ON are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Again let $\angle AON = \angle NOB = q$ [Law of reflection]

$\therefore \cos\theta = 1.\frac{1}{\sqrt{3}} + 0 + 0$ $\quad [\because \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2]$

Let l, m, n be Dc's of reflected ray OB.

$\cos\theta = \frac{1}{\sqrt{3}}.l + \left(-\frac{1}{\sqrt{3}}\right)m + \frac{1}{\sqrt{3}}n$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{l}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \quad \Rightarrow \quad l - m + n = 1 \quad \ldots (i)$

Also, $\cos 2\theta = 1.\, l + 0\,.\, m + 0\,.\, n$

$\Rightarrow 2\cos^2\theta - 1 = l$

$\Rightarrow 2 \times \frac{1}{3} - 1 = l \quad \Rightarrow \quad l = \frac{2-3}{3} = -\frac{1}{3}$

Putting in (i), we get $m - n = -\frac{4}{3} \quad \ldots (ii)$

Also $l^2 + m^2 + n^2 = 1$

$m^2 + n^2 = 1 - \left(-\frac{1}{3}\right)^2 = \frac{8}{9} \quad \ldots (iii)$

(ii) & $(iii) \Rightarrow m = -\frac{2}{3}$ and $n = \frac{2}{3}$, Hence, direction cosines of reflected ray are $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.

Q.2. Find the cartesian as well as vector equations of the planes through the intersection of planes $\vec{r}.\,(2\hat{i} + 6\hat{j}) + 12 = 0$ **and** $\vec{r}.(3\hat{i} - \hat{j} + 4\hat{k}) = 0$ **, which are at a unit distance from the origin.**

Ans.

The equation of the plane passing through the intersection of the planes

$\vec{r}.(2\hat{i}+6\hat{j})+12=0$ and $\vec{r}.(3\hat{i}-\hat{j}+4\hat{k})=0$ is

$\Rightarrow \vec{r}.\{(2+3\lambda)\hat{i}+(6-\lambda)\hat{j}+4\lambda\hat{k}\}+12=0 \qquad ...(i)$

The planes are at a unit distance from origin. Therefore, length of the perpendicular from the origin to the plane (i) = 1 unit.

$\therefore \frac{12}{\sqrt{(2+3\lambda)^2+(6-\lambda)^2+16\lambda^2}}=1$

$\Rightarrow 144 = (2+3\lambda)^2 + (6-\lambda)^2 + 16\lambda^2 \qquad \Rightarrow 144 = 40 + 26\lambda^2 \qquad \Rightarrow 26\lambda^2 = 104$

$\Rightarrow \lambda^2 = 4 \qquad \Rightarrow \lambda = \pm 2$

Putting the values of λ in equation (i), we get

$\vec{r}.(8\hat{i}+4\hat{j}+8\hat{k})+12=0$ and $\vec{r}.(-4\hat{i}+8\hat{j}-8\hat{k})+12=0$

which are the equations of the required planes. These equations can also be written as

$\vec{r}.(2\hat{i}+\hat{j}+2\hat{k})+3=0$ and $\vec{r}.(-\hat{i}+2\hat{j}-2\hat{k})+3=0$.

The above equations can be written in cartesian form as follows:

$2x + y + 2z + 3 = 0$ and $-x + 2y - 2z + 3 = 0$

Q.3. A plane meets the coordinate axes in *A, B, C,* such that the centroid of the triangle *ABC* is the point (α, β, γ). Show that the equation of the plane is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3.$

Ans.

Let the equation of required plane be

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ...(i)

Then the coordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. So, the centroid of triangle ΔABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$. But the coordinates of the centroid are (α, β, γ) as given in problem.

$\alpha=\frac{a}{3}$, $\beta=\frac{b}{3}$, and $\gamma=\frac{c}{3} \Rightarrow a=3\alpha, b=3\beta, c=3\gamma$.

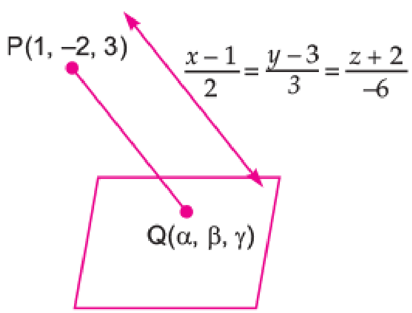
Substituting the values of a, b and c in equation (i), we get the required equation of the plane as follows

$\frac{x}{3\alpha}+\frac{y}{3\beta}+\frac{z}{3\gamma}=1 \quad \Rightarrow \quad \frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3.$

Q.4. Find the distance of the point (1, –2, 3) from the plane *x* – *y* + *z* = 5, measured parallel to the line.

$$\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+2}{-6}$$

Ans.



Let $Q(\alpha, \beta, \gamma)$ be the point on the given plane

$x-y+z=5$ ...(i)

Since PQ is parallel to given line

$\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+2}{-6}$ ...(ii) where $P(1, -2, 3)$ is the given point.

$\because$ PQ is parallel to given line (ii).

$\therefore \overrightarrow{PQ} \parallel \vec{b}$ (parallel vector of line).

$\Rightarrow \frac{\alpha-1}{2}=\frac{\beta+2}{3}=\frac{\gamma-3}{-6}=\lambda$

$\Rightarrow \alpha = 2\lambda + 1, \beta = 3\lambda - 2, \gamma = - 6\lambda + 3$

Now, $\because Q(\alpha, \beta, \gamma)$ lie on plane (i)

$\alpha - \beta + \gamma = 5$

$2\lambda + 1 - 3\lambda + 2 - 6\lambda + 3 = 5$

$-7\lambda + 6 = 5 \qquad \Rightarrow -7\lambda = -1$

$\lambda = \frac{1}{7}$

$\alpha = 2 \times \frac{1}{7} + 1 = \frac{9}{7}; \beta = 3 \times \frac{1}{7} - 2 = - \frac{11}{7}$ and $\gamma = - 6 \times \frac{1}{7} + 3 = \frac{15}{7}$

Therefore required distance

$$\mathrm{PQ} = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(- \frac{11}{7} + 2\right)^2 + \left(\frac{15}{7} - 3\right)^2} \text{ unit.}$$
$$= \sqrt{\frac{4}{49} + \frac{9}{49} \frac{36}{49}} = \sqrt{1} = 1$$