RD Sharma Solutions Class 12 Maths Chapter 15 Ex 15.1

Mean Value Theorems Ex 15.1 Q1(i)

$f(x) = 3 + (x-2)^{\frac{2}{3}}$ on $[1,3]$

Differentiating it with respect to x,

$$f'(x) = \frac{2}{3} \times \frac{1}{(x-2)^{\frac{1}{3}}}$$

Clearly, $\lim\limits_{x \to 2} = \frac{2}{3} \times \frac{1}{(x-2)^{\frac{1}{3}}}$

Thus, $f(x)$ is not differentiable at $x = 2 \in (1,3)$

Hene, Rolle's theorem is not applicable for $f(x)$ in $x \in [1,3]$.

Mean Value Theorems Ex 15.1 Q1(ii)

Here, $f(x) = [x]$ and $x \in [-1,1]$, at $n = 1$

$$\begin{aligned} \text{LHL} &= \lim_{x \to (1-h)} [x] \\ &= \lim_{h \to 0} [1-h] \\ &= 0 \\ \text{RHL} &= \lim_{x \to (1+h)} [x] \\ &= \lim_{h \to 0} [1+h] \\ &= 1 \\ &\text{LHL} \neq RHL \end{aligned}$$

So, $f(x)$ is not continuous at $1 \in [-1,1]$

Hence, rolle's theorem is not applicable on $f(x)$ in $[-1,1]$.

Mean Value Theorems Ex 15.1 Q1(iii)

Here, $f(x) = \sin\left(\frac{1}{x}\right)$, $x \in [-1,1]$, at $n = 0$

$$\text{LHS} = \lim_{x \to (0-h)} \sin\left(\frac{1}{x}\right)$$
$$= \lim_{h \to 0} \sin\left(\frac{1}{0-h}\right)$$
$$= \lim_{h \to 0} \sin\left(\frac{-1}{h}\right)$$
$$= -\lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$
$$= -k \qquad \left[\text{Let } \lim_{h \to 0} \sin\left(\frac{1}{h}\right) = k \text{ as } k \in [-1,1]\right]$$

$$\text{RHS} = \lim_{x \to (0+h)} \sin\left(\frac{1}{x}\right)$$
$$= \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$
$$= k$$

$\Rightarrow$ LHS $\neq$ RHS

$\Rightarrow$ $f(x)$ is not continuous at $n = 0$

So, rolle's theorem is not applicable on $f(x)$ in $[-1,1]$

Mean Value Theorems Ex 15.1 Q1(iv)

Here, $f(x) = 2x^2 - 5x + 3$ on $[1,3]$

$f(x)$ is continuous in $[1,3]$ and $f(x)$ is differentiable is $(1,3)$ since it is a polynomial function.

Now,

$$f(x) = 2x^2 - 5x + 3$$
$$f(1) = 3(1)^2 - 5(1) + 3$$
$$= 2 - 5 + 3$$
$$f(1) = 0 \qquad \text{---(i)}$$
$$f(3) = 2(3)^2 - 5(3) + 3$$
$$= 18 - 15 + 3$$
$$f(3) = 6 \qquad \text{---(ii)}$$

From equation (i) and (ii),

$$f(1) \neq f(3)$$

So, rolle's theorem is not applicable on $f(x)$ in $[1,3]$.

Mean Value Theorems Ex 15.1 Q1(v)

Here, $f(x) = x^{\frac{2}{3}}$ on $[-1,1]$

$$f'(x) = \frac{2}{3x^{\frac{1}{3}}}$$

$$f'(0) = \frac{2}{3(0)^{\frac{1}{3}}}$$

$$f'(0) = \infty$$

So, $f'(x)$ does not exist at $x = 0 \in (-1,1)$

$\Rightarrow$ $f(x)$ is not differentiatable in $x \in (-1,1)$

So, rolle's theorem is not applicable on $f(x)$ in $[-1,1]$.

Mean Value Theorems Ex 15.1 Q1(vi)

Here, $f(x) = \begin{cases} -4x+5, & 0 \le x \le 1 \\ 2x-3, & 1 < x \le 2 \end{cases}$

For $n = 1$

LHS $= \lim\limits_{x \to (1-h)} (-4x+5)$

$= \lim\limits_{h \to 0} [-4(1-h)+5]$

$= -4+5$

LHS $= 1$

RHS $= \lim\limits_{x \to (1+h)} (2x-3)$

$= \lim\limits_{h \to 0} [2(1+h)-3]$

$= 2-3$

RHS $= -1$

So, LHS $\neq$ RHS

$\Rightarrow$ $f(x)$ is not contiuuous at $x = 1 \in [0,2]$

$\Rightarrow$ Rolle's theorem is not applicable on $f(x)$ in $[0,2]$.

Mean Value Theorems Ex 15.1 Q2(i)

Here,

$$f(x) = x^2 - 8x + 12 \text{ on } [2,6]$$

$f(x)$ is continuous is $[2,6]$ and differentiable is $(2,6)$ as it is a polynomial function

And $f(2) = (2)^2 - 8(2) + 12 = 0$

$f(6) = (6)^2 - 8(6) + 12 = 0$

$\Rightarrow$ $f(2) = f(6)$

So, Rolle's theorem is applicable, therefore we show have $f'(c) = 0$ such that $c \in (2,6)$

So, $f(x) = x^2 - 8x + 12$

$\Rightarrow$ $f'(x) = 2x - 8$

So, $f'(c) = 0$

$2c - 8 = 0$

$c = 4 \in (2,6)$

Therefore, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(ii)

The given function is $f(x) = x^2 - 4x + 3$

f, being a polynomial function, is continuous in $[1,4]$ and is differentiable in $(1,4)$ whose derivative is $2x - 4$.

$f(1) = 1^2 - 4\times1 + 3 = 0$, $f(4) = 4^2 - 4\times4 + 3 = 3$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{3 - (0)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point $c \in (1,4)$ such that $f'(c) = 1$

$f'(c) = 1$

$\Rightarrow 2c - 4 = 1$

$\Rightarrow c = \frac{5}{2}$, where $c = \frac{5}{2} \in (1, 4)$

Hence, Mean Value Theorem is verified for the given function

Mean Value Theorems Ex 15.1 Q2(iii)

Here,

$$f(x) = (x-1)(x-2)^2 \text{ on } (1,2)$$

$f(x)$ is cantinuous is $[1,2]$ and differentiable is $(1,2)$ since it is a polynomial function.

And $f(1) = (1-1)(1-2)^2 = 0$

$$f(2) = (2-1)(2-2)^2 = 0$$

$\Rightarrow \quad f(1) = f(2)$

So, Rolle's theorem is applicable on $f(x)$ in $[1,2]$, therefore, there exist a $c \in (1,2)$ such that $f'(c) = 0$

Now,

$$f(x) = (x-1)(x-2)^2$$

$$f'(x) = (x-1) \times 2(x-2) + (x-2)^2$$

$$f'(x) = (x-2)(3x-4)$$

So, $f'(c) = 0$

$$(c-2)(3c-4) = 0$$

$\Rightarrow \quad c = 2 \text{ or } c = \frac{4}{3} \in (1,2)$

Thus, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(iv)

Here,

$$f(x) = x(x-1)^2 \text{ on } [0,1]$$

$f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ as it is a polynomial function.

Now,

$$f(0) = 0(0-1)^2 = 0$$

$$f(1) = 1(1-1)^2 = 0$$

$$\Rightarrow \quad f(0) = f(1)$$

So, Rolle's theorem is applicable on $f(x)$ in $[0,1]$ therefore, we should show that there exist a $c \in (0,1)$ such that $f'(c) = 0$

Now,

$$f(x) = x(x-1)^2$$

$$f'(x) = (x-1)^2 + x \times 2(x-1)$$

$$= (x-1)(x-1+2x)$$

$$f'(x) = (x-1)(3x-1)$$

So, $\quad f'(c) = 0$

$$(c-1)(3c-1) = 0$$

$$\Rightarrow \quad c = 1 \text{ or } c = \frac{1}{3} \in (0,1)$$

Thus, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(v)

Here,

$$f(x) = (x^2 - 1)(x - 2) \text{ on } [-1, 2]$$

$f(x)$ is continuous is $[-1,2]$ and differentiable in $(-1,2)$ as it is a polynomial functions.

Now,

$$f(-1) = (1-1)(-1-2) = 0$$

$$f(2) = (4-1)(2-2) = 0$$

$$\Rightarrow \quad f(-1) = f(2)$$

So, Rolle's theorem is applicable on $f(x)$ is $[-1,2]$ therefore, we have to show that there exist a $c \in (-1,2)$ such that $f'(c) = 0$

Now,

$$f(x) = (x^2 - 1)(x - 2)$$

$$f'(x) = 2x(x-2) + (x^2 - 1)$$

$$= 2x^2 - 4 + x^2 - 1$$

$$f'(x) = 3x^2 - 5$$

Now,

$$f'(c) = 0$$

$$\Rightarrow \quad 3x^2 - 5 = 0$$

$$\Rightarrow \quad x = -\sqrt{\frac{5}{3}} \text{ or } x = \sqrt{\frac{5}{3}} \in (-1,2)$$

Thus, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(vi)

Here, $f(x) = x(x-4)^2$ on $[0,4]$

$f(x)$ is continuous is $[0,4]$ and differentiable is $(0,4)$ since $f(x)$ is a polynomial function.

Now,

$$f(x) = x(x-4)^2$$

$$f(0) = 0(0-4)^2$$

$$f(0) = 0 \qquad \text{---(i)}$$

$$f(4) = 4(4-4)^2$$

$$f(4) = 0 \qquad \text{---(ii)}$$

From equation (i) and (ii),

$$f(0) = f(4)$$

So, Rolle's theorem is applicable, therefore, we have to show that $f'(c) = 0$ for $c \in (0,4)$

$$f'(x) = x \times 2(x-4) + (x-4)^2$$

$$= 2x^2 - 8x + x^2 + 16 - 8x$$

So, $f'(c) = 3c^2 - 16c + 16$

$$0 = 3c^2 - 12c - 4c + 16$$

$$0 = 3c(c-4) - 4(c-4)$$

$$0 = (c-4)(3c-4)$$

$\Rightarrow \quad c = 4$ or $c = \frac{4}{3} \in (0,4)$

So, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(vii)

Here, $f(x) = x(x-2)^2$ on $[0,2]$

$f(x)$ is continuous is $[0,2]$ and differentiable is $(0,2)$

as it is a polynomial function.

And $\quad f(0) = 0(0-2)^2 = 0$

$$f(2) = 2(2-2)^2 = 0$$

$$\Rightarrow \quad f(0) = f(2)$$

So, Rolle's theorem is applicable on $f(x)$ is $[0,2]$, therefore,

we have to show that $f'(c) = 0$ as $c \in (0,2)$

$$f(x) = x(x-2)^2$$

$$f'(x) = x \times 2(x-2) + (x-2)$$

$$f'(x) = 2x(x-2) + (x-2)$$

$$\Rightarrow \quad f'(c) = 0$$

$$2c(c-2) + (c-2) = 0$$

$$(c-2)(2c+1) = 0$$

$$c = 2 \text{ or } c = -\frac{1}{2}$$

$$\Rightarrow \quad c = 2 \in (0,2)$$

So, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q2(viii)

Here, $f(x) = x^2 + 5x + 6$ on $[-3, -2]$

$f(x)$ is continuous is $[-3, -2]$ and $f(x)$ is differentiable is $(-3, -2)$ since it is a polynomial function.

Now,

$$f(x) = x^2 + 5x + 6$$

$$f(-3) = (-3)^2 + 5(-3) + 6$$

$$= 9 - 15 + 6$$

$$f(-3) = 0 \qquad \text{---(i)}$$

$$f(-2) = (-2)^2 + 5(-2) + 6$$

$$= 4 - 10 + 6$$

$$f(-2) = 20 \qquad \text{---(ii)}$$

From equation (i) and (ii),

$$f(-3) = f(-2)$$

So, Rolle's theorem is applicable is $[-3, -2]$, we have to show that $f'(c) = 0$ as $c \in (-3, -2)$.

Now,

$$f(x) = x^2 + 5x + 6$$

$$f'(x) = 2x + 5$$

$$\Rightarrow \qquad f'(c) = 0$$

$$2c + 5 = 0$$

$$c = \frac{-5}{2} \in (-3, -2)$$

So, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(i)

Here,

$$f(x) = \cos 2\left(x - \frac{\pi}{4}\right) \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that cosine function is continuous and differentiable

every where, so $f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and differentiable is $\left[0, \frac{\pi}{2}\right]$.

Now,

$$f(0) = \cos 2\left(0 - \frac{\pi}{4}\right) = 0$$

$$f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable.

Hence, there must exists a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f'(x) = -\sin 2\left(x - \frac{\pi}{4}\right) \times 2$$

$$f'(x) = -2\sin\left(2x - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad -2\sin\left(2c - \frac{\pi}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(2c - \frac{\pi}{2}\right) = \sin 0$$

$$\Rightarrow \quad 2c - \frac{\pi}{2} = 0$$

$$c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(ii)

Here,

$$f(x) = \sin 2x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that $\sin x$ is a continuous and differentiable every where. So,

$f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$ and differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin 0 = 0$$

$$f\left(\frac{\pi}{2}\right) = \sin \pi = 0$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so, there must exist a $c \in \left(0, \frac{\pi}{2}\right)$

such that $f'(c) = 0$

Now,

$$f'(x) = 2\cos 2x$$

$$f'(c) = 2\cos 2c = 0$$

$$\Rightarrow \quad \cos 2c = 0$$

$$\Rightarrow \quad 2c = \frac{\pi}{2}$$

$$\Rightarrow \quad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{4}\right)$$

Thus, Rolle's theorem verified.

Mean Value Theorems Ex 15.1 Q3(iii)

Here,

$$f(x) = \cos 2x \text{ on } \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$$

We know that $\cos x$ is a continuous and differentiable every where. So, $f(x)$ is continuous in $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$ and differentiable is $\left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$.

Now, $f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

$$f\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow \quad f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$$

So, Rolle's theorem is applicable, so, there must exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$

Now,

$$f'(x) = 2 \sin 2x$$

$$f'(c) = 2 \sin 2c = 0$$

$$\Rightarrow \quad \sin 2c = 0$$

$$\Rightarrow \quad 2c = 0$$

$$\Rightarrow \quad c = 0 \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$$

Thus, Rolle's theorem verified.

Mean Value Theorems Ex 15.1 Q3(iv)

Here,

$$f(x) = e^x \times \sin x \quad \text{on } [0, \pi]$$

We know that since and expential function are continuous and differentiable every where so, $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = e^0 \sin 0 = 0$$

$$f(\pi) = e^\pi \sin \pi = 0$$

$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = e^x \sin x$$

$$f'(x) = e^x \cos x + e^x \sin x$$

Now, $f'(c) = 0$

$$e^c (\cos c + \sin c) = 0$$

$$\Rightarrow \quad e^c = 0 \text{ or } \cos c = -\sin c$$

$$\Rightarrow \quad e^c = 0 \text{ gives no value of } c \text{ or } \tan c = -1$$

$$\Rightarrow \quad \tan c = \tan\left(\pi - \frac{\pi}{4}\right)$$

$$c = \frac{3\pi}{4} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(v)

Here,

$$f(x) = e^x \cos x \text{ on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

We know that expontial and cosine function are continuous and differentiable every where so, $f(x)$ is continuous is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and differentiable is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now,

$$f\left(-\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} \cos\left(-\frac{\pi}{2}\right) = 0$$

$$f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}\right) = 0$$

$$\Rightarrow \quad f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = e^x \cos x$$

$$f'(x) = -e^x \sin x + e^x \cos x$$

So, $\quad f'(c) = 0$

$$e^c(-\sin c + \cos c) = 0$$

$\Rightarrow \quad e^c = 0$ gives no value of c

$$\Rightarrow \quad -\sin c + \cos c = 0$$

$$\Rightarrow \quad \tan c = 1$$

$$\Rightarrow \quad c = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(vi)

Here,

$$f(x) = \cos 2x \text{ on } [0, \pi]$$

We know that, cosine function is contiuous and differentiable every where, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = \cos 0 = 1$$

$$f(\pi) = \cos(2\pi) = 1$$

$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = \cos 2x$$

$$f'(x) = -2\sin 2x$$

So, $\quad f'(c) = 0$

$$\Rightarrow \quad -2\sin 2c = 0$$

$$\Rightarrow \quad \sin 2c = 0$$

$$\Rightarrow \quad 2c = 0 \quad \text{or} \quad 2c = \pi$$

$$\Rightarrow \quad c = 0 \quad \text{or} \quad c = \frac{\pi}{2} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(vii)

$f(x)=\frac{\sin x}{e^x}$ *on* $x \in [0, \pi]$

We know that, exponential and sine *both functions are continuous and differentiabl* *every where, so* $f(x)$ *is continuous is* $[0, \pi]$ *and differentiable is* $[0, \pi]$

Now,

$$f(0)=\frac{\sin 0}{e^0}=0$$

$$f(\pi)=\frac{\sin \pi}{e^\pi}=0$$

$\Rightarrow f(0)=f(\pi)$

Since Rolle's theorem applicable, therefore there must exist a point $c \in [0, \pi]$ such that $f'(c)=0$

Now,

$$f(x)=\frac{\sin x}{e^x}$$

$$\Rightarrow f'(x)=\frac{e^x(\cos x)-e^x(\sin x)}{\left(e^x\right)^2}$$

Now,

$$f'(c)=0$$

$$\Rightarrow e^c(\cos c-\sin c)=0$$

$$\Rightarrow e^c \neq 0 \text{ and } \cos c-\sin c=0$$

$$\Rightarrow \tan c=1$$

$$c=\frac{\pi}{4} \in [0,\pi]$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(viii)

Here,

$$f(x) = \sin 3x \text{ on } [0, \pi]$$

We know that, sine function is continuous and differentiable every where. So, $f(x)$ is continuous is $(0, \pi)$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = \sin 0 = 0$$

$$f(\pi) = \sin 3\pi = 0$$

$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exists a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin 3x$$

$$f'(x) = 3\cos 3x$$

Now,

$$f'(c) = 0$$

$$\Rightarrow \quad 3\cos 3x = 0$$

$$\Rightarrow \quad \cos 3x = 0$$

$$\Rightarrow \quad 3x = \frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{\pi}{6} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(ix)

Here,

$$f(x) = e^{1-x^2} \text{ on } [-1, 1]$$

We know that, exponential function is continuous and differentiable every where. So, $f(x)$ is continuous is $[-1,1]$ and differentiable is $(-1,1)$.

Now,

$$f(-1) = e^{1-1} = 1$$

$$f(1) = e^{1-1} = 1$$

$$\Rightarrow \quad f(-1) = 1$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (-1,1)$ such that $f'(c) = 0$.

Now,

$$f(x) = e^{1-x^2}$$

$$f'(x) = e^{1-x^2}(-2x)$$

Now,

$$f'(c) = 0$$

$$-2ce^{1-c^2} = 0$$

$$\Rightarrow \quad c = 0 \quad \text{or} \quad e^{1-c^2} = 0$$

$$\Rightarrow \quad c = 0 \in (-1, 1)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(x)

Here,

$$f(x) = \log\left(x^2 + 2\right) - \log 3 \text{ on } [-1,1]$$

We know that, logarithmic function is continuous and differentiable is its domain, so $f(x)$ is continuous is $[-1,1]$ and differentiable is $(-1,1)$.

Now,

$$f(-1) = \log(1+2) - \log 3 = 0$$

$$f(1) = \log(1+2) - \log 3 = 0$$

$$\Rightarrow \quad f(-1) = f(1)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (-1,1)$ such that $f'(c) = 0$.

Now,

$$f(x) = \log\left(x^2 + 2\right) - \log 3$$

$$f'(x) = \frac{(2x)}{x^2 + 2}$$

Now,

$$f'(c) = 0$$

$$\frac{2c}{c^2 + 2} = 0$$

$$\Rightarrow \quad c = 0 \in (-1,1)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xi)

Here,

$$f(x) = \sin x + \cos x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that $\sin x$ and $\cos x$ are continuous and differentiable every where, so $f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin 0 + \cos c0 = 1$$

$$f\left(\frac{\pi}{2}\right) = \frac{\sin \pi}{2} + \frac{\cos \pi}{2} = 1$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin x + \cos x$$

$$f'(x) = \cos x - \sin x$$

Now,

$$f'(c) = 0$$

$$\cos c - \sin c = 0$$

$$\Rightarrow \quad \tan c = 1$$

$$\Rightarrow \quad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xii)

Here,

$$f(x) = 2\sin x + \sin 2x \text{ on } [0, \pi]$$

We know that sine function is continuous and differentiable every where, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = 2\sin 0 + \sin 0 = 0$$

$$f(\pi) = 2\sin \pi + \sin 2\pi = 0$$

$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = 2\sin x + \sin 2x$$

$$f'(x) = 2\cos x + 2\cos 2x$$

Now,

$$f'(c) = 0$$

$$2\cos c + 2\cos 2c = 0$$

$$\Rightarrow \quad 2\left(\cos c + 2\cos^2 c - 1\right) = 0$$

$$\Rightarrow \quad \left(2\cos^2 + 2\cos c - \cos c - 1\right) = 0$$

$$\Rightarrow \quad (2\cos c - 1)(\cos c + 1) = 0$$

$$\Rightarrow \quad \cos c = \frac{1}{2}, \quad \cos c = -1$$

$$\Rightarrow \quad \tan c = 1$$

$$c = \frac{\pi}{3} \in (0, \pi), \; c = \pi$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xiii)

Here,

$$f(x) = \frac{x}{2} - \sin\frac{\pi x}{6} \text{ on } [-1, 0]$$

We know that sine function is continuous and differentiable every where, so $f(x)$ is continuous is $[-1, 0]$ and differentiable is $(-1, 0)$.

Now,

$$f(-1) = \frac{-1}{2} - \sin\left(-\frac{\pi}{6}\right)$$

$$= -\frac{1}{2} + \sin\frac{\pi}{6}$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$f(-1) = 0 \qquad \text{---(i)}$$

And $$f(0) = 0 - \sin 0$$

$$f(0) = 0 \qquad \text{---(ii)}$$

From equation (i) and (ii),

$$f(-1) = f\begin{pmatrix}0\\ \end{pmatrix}$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (-1, 0)$ such that $f'(c) = 0$.

Now,

$$f(x) = \frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)$$

$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi x}{6}\right)$$

Now,

$$f'(c) = 0$$

$$\frac{1}{2} - \frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = 0$$

$$\Rightarrow \quad -\frac{\pi}{6}\cos\left(\frac{\pi c}{6}\right) = -\frac{1}{2}$$

$$\Rightarrow \quad \cos\left(\frac{\pi c}{6}\right) = 3\pi$$

$$\Rightarrow \quad \frac{\pi c}{6} = \cos^{-1}\left(\frac{66}{7}\right)$$

$$\Rightarrow \quad c = \frac{6}{\pi}\cos^{-1}\left(\frac{66}{7}\right)$$

$$\Rightarrow \quad c = \frac{21}{11}\cos^{-1}\left(\frac{66}{7}\right)$$

$$\Rightarrow \quad c \in \left(-\frac{21}{11}, \frac{21}{11}\right) \qquad \left[\text{since, } \cos^{-1} x \in [-1, 1]\right]$$

$$\Rightarrow \quad c \in (-1.9, 1.9)$$

$$\Rightarrow \quad c \in (-1, 0)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xiv)

Here,

$$f(x) = \frac{6x}{\pi} - 4\sin^2 x \text{ on } \left[0, \frac{\pi}{6}\right]$$

We know that sine and its square function is continuous and differentiable every where, s

$f(x)$ is continuous is $\left[0, \frac{\pi}{6}\right]$ and differentiable is $\left(0, \frac{\pi}{6}\right)$.

Now,

$$f(0) = 0 - 0 = 0$$

$$f\left(\frac{\pi}{6}\right) = 1 - 1 = 0$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{6}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{6}\right)$ such

that $f'(c) = 0$.

Now,

$$f(x) = \frac{6x}{\pi} - 4\sin^2 x$$

$$f'(x) = \frac{6}{\pi} - 8\sin x \cos x$$

$$f'(x) = \frac{6}{\pi} - 4\sin 2x$$

Now,

$$f'(c) = 0$$

$$\frac{6}{\pi} - 4\sin 2c = 0$$

$$\Rightarrow \quad 4\sin 2c = \frac{6}{\pi}$$

$$\Rightarrow \quad \sin 2c = \frac{3}{2\pi}$$

$$\Rightarrow \quad 2c = \sin^{-1}\left(\frac{21}{44}\right)$$

$$\Rightarrow \quad c = \frac{1}{2}\sin^{-1}\left(\frac{21}{44}\right)$$

$$\Rightarrow \quad c \in \left(-\frac{1}{2}, \frac{1}{2}\right) \qquad \left[\text{since, } \sin^{-1} x \in [-1, 1]\right]$$

$$\Rightarrow \quad c \in \left(0, \frac{11}{21}\right)$$

$$\Rightarrow \quad c \in \left(0, \frac{\pi}{6}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xv)

Here,

$$f(x) = 4^{\sin x} \text{ on } [0, \pi]$$

We know that exponential and $\sin x$ both are continuous and differentiable, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = 4^{\sin 0} = 4^0 = 1$$
$$f(\pi) = 4^{\sin \pi} = 4^0 = 1$$
$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = 4^{\sin x}$$
$$f'(x) = 4^{\sin x} \log 4 \times \cos x$$

Now,

$$f'(c) = 0$$
$$4^{\sin c} \times \cos x c \log 4 = 0$$
$$\Rightarrow \quad \cos c = 0$$
$$\Rightarrow \quad c = \frac{\pi}{2} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvi)

Here,

$$f(x) = x^2 - 5x + 4 \text{ on } [1, 4]$$

$f(x)$ is continuous and differentiable as it is a polynomial funciton.

Now,

$$f(1) = (1)^2 - 5(1) + 4 = 0$$
$$f(4) = (4)^2 - 5(4) + 4 = 0$$
$$\Rightarrow \quad f(1) = f(4)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (1, 4)$ such that $f'(c) = 0$.

Now,

$$f(x) = x^2 - 5x + 4$$
$$f'(x) = 2x - 5$$

So,

$$f'(c) = 0$$
$$\Rightarrow \quad 2c - 5 = 0$$
$$\Rightarrow \quad c = \frac{5}{2} \in (1, 4)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvii)

Here,

$$f(x) = \sin^4 x + \cos^4 x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that *sine* and *cosine* function are differentiable and continuous.

So, $f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and it is differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin^4(0) + \cos^4(0) = 1$$

$$f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right) = 1$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{2}\right)$ such

that $f'(c) = 0$.

Now,

$$f(x) = \sin^4 x + \cos^4 x$$

$$f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$$

$$= -2(2\sin x \cos x)\left(\cos^2 x - \cos^2 x\right)$$

$$= -2\sin 2x \cos 2x$$

$$f'(x) = -\sin 4x$$

Now,

$$f'(c) = 0$$

$$-\sin 4x = 0$$

$$\sin 4x = 0$$

$$\Rightarrow \quad 4x = 0 \quad \text{or} \quad 4x = \pi$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvii)

Since trigonometric functions are differentiable and continuous,
the given function, $f(x) = \sin x - \sin 2x$ is also continuous and differentiable.

Now $f(0) = \sin 0 - \sin 2 \times 0 = 0$

and

$f(\pi) = \sin \pi - \sin 2 \times \pi = 0$

$\Rightarrow f(0) = f(\pi)$

Thus, $f(x)$ satisfies conditions of the Rolle's Theorem on $[0, \pi]$.

Therefore, there exists $c \in [0, \pi]$ such that $f'(c) = 0$

Now $f(x) = \sin x - \sin 2x$

$\Rightarrow f'(x) = \cos x - 2\cos 2x = 0$

$\Rightarrow \cos x = 2\cos 2x$

$\Rightarrow \cos x = 2(2\cos^2 x - 1)$

$\Rightarrow \cos x = 4\cos^2 x - 2$

$\Rightarrow 4\cos^2 x - \cos x - 2 = 0$

$$\Rightarrow \cos x = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$$

$\Rightarrow x = \cos^{-1}(0.8431)$ or $\cos^{-1}(-0.5931)$

$\Rightarrow x = \cos^{-1}(0.8431)$ or $180^\circ - \cos^{-1}(0.5931)$ $\quad [\because \cos^{-1}(-x) = \pi - \cos^{-1}(x)]$

$\Rightarrow x = 32^\circ 32'$ or $x = 126^\circ 23'$

Both $32^\circ 32'$ and $126^\circ 23' \in [0, \pi]$ such that $f'(c) = 0$.

Hence Rolle's Theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xviii)

Since trigonometric functions are differentiable and continuous, the given function, $f(x) = \sin x - \sin 2x$ is also continuous and differentiable.

Now $f(0) = \sin 0 - \sin 2 \times 0 = 0$

and

$f(\pi) = \sin \pi - \sin 2 \times \pi = 0$

$\Rightarrow f(0) = f(\pi)$

Thus, $f(x)$ satisfies conditions of the Rolle's Theorem on $[0, \pi]$.

Therefore, there exists $c \in [0, \pi]$ such that $f'(c) = 0$

Now $f(x) = \sin x - \sin 2x$

$\Rightarrow f'(x) = \cos x - 2\cos 2x = 0$

$\Rightarrow \cos x = 2\cos 2x$

$\Rightarrow \cos x = 2(2\cos^2 x - 1)$

$\Rightarrow \cos x = 4\cos^2 x - 2$

$\Rightarrow 4\cos^2 x - \cos x - 2 = 0$

$\Rightarrow \cos x = \dfrac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } \cos^{-1}(-0.5931)$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } 180^\circ - \cos^{-1}(0.5931) \quad [\because \cos^{-1}(-x) = \pi - \cos^{-1}(x)]$

$\Rightarrow x = 32^\circ 32' \text{ or } x = 126^\circ 23'$

Both $32^\circ 32'$ and $126^\circ 23' \in [0, \pi]$ such that $f'(c) = 0$.

Hence Rolle's Theorem is verified.

Mean Value Theorems Ex 15.1 Q7

Let $f(x) = 16 - x^2$, then $f'(x) = -2x$

$f(x)$ is continuous on $[-1, 1]$ because it is a polynomial function.

Also $f(-1) = 16 - (-1)^2 = 15$

$f(1) = 16 - (1)^2 = 15$

$f(-1) = f(1)$

There exists a $c \in [-1, 1]$ such that $f'(c) = 0$

$\Rightarrow -2c = 0$

$\Rightarrow \quad c = 0$

Thus, at $0 \in [-1, 1]$ the tangent is parallel to the x-axis.

Mean Value Theorems Ex 15.1 Q8(i)

Let $f(x) = x^2$, then $f'(x) = 2x$

$f(x)$ is continuous on $[-2,2]$ because it is a polynomial function.

$f(x)$ is differentiable on $(-2,2)$ as it is a polynomial function.

Also $f(-2) = (-2)^2 = 4$

$f(2) = 2^2 = 4$

$\Rightarrow \quad f(-2) = f(2)$

$\therefore$ There exists $c \in (-2,2)$ such that $f'(c) = 0$

$\Rightarrow \quad 2c = 0$

$\Rightarrow \quad c = 0$

Thus, at $0 \in [-2,2]$ the tangent is parallel to the x-axis.

x = 0, then y = 0

Therefore, the point is (0, 0)

Mean Value Theorems Ex 15.1 Q8(ii)

Let $f(x) = e^{1-x^2}$ on $[-1,1]$

Since, $f(x)$ is a composition of two continuous functions, it is continuous on $[-1,1]$

Also $f(x) = -2xe^{1-x^2}$

$f(2) = 2^2 = 4$

$\therefore \quad f'(x)$ exists for every value of x in $(-1,1)$

$\Rightarrow \quad f(x)$ is differentiable on $(-1,1)$

By rolle's theorem, there exists $c \in (-1,1)$ such that $f'(c) = 0$

$\Rightarrow \quad -2ce^{1-c^2} = 0$

$\Rightarrow \quad c = 0$

Thus, at $c = 0 \in [-1,1]$ the tangent is parallel to the x-axis.

x = 0, then y = e

Therefore, the point is (0, e)

Mean Value Theorems Ex 15.1 Q8(iii)

Let $f(x) = 12(x+1)(x-2)$

Since, $f(x)$ is a polynomial function, it is continuous on $[-1,2]$ and differentiable on $(-1,2)$

Also $f'(x) = 12[(x-2)+(x+1)] = 12[2x-1]$

By rolle's theorem, there exists $c \in (-1,2)$ such that $f'(c) = 0$

$\Rightarrow \quad 12(2c-1) = 0$

$\Rightarrow \quad c = \frac{1}{2}$

Thus, at $c = \frac{1}{2} \in (-1,2)$ the tangent to $y = 12(x+1)(x-2)$ is parallel to x-axis

Mean Value Theorems Ex 15.1 Q9

It is given that $f:[-5,5] \to \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on $[-5, 5]$.

(b) f is differentiable on $(-5, 5)$.

Therefore, by the Mean Value Theorem, there exists c □ $(-5, 5)$ such that

$$f'(c) = \frac{f(5)-f(-5)}{5-(-5)}$$
$$\Rightarrow 10f'(c) = f(5)-f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c) \neq 0$$
$$\Rightarrow 10f'(c) \neq 0$$
$$\Rightarrow f(5)-f(-5) \neq 0$$
$$\Rightarrow f(5) \neq f(-5)$$

Hence, proved.

Mean Value Theorems Ex 15.1 Q10

By Rolle's Theorem, for a function $f:[a, b]\to \mathbf{R}$, if

(a) f is continuous on $[a, b]$

(b) f is differentiable on (a, b)

(c) $f(a) = f(b)$

then, there exists some $c \in (a, b)$ such that $f'(c) = 0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any the three conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$f(x)$ is not continuous in $[5, 9]$.

Also, $f(5) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore f(5) \neq f(9)$

The differentiability of f in $(5, 9)$ is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

f is not differentiable in $(5, 9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

f is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.