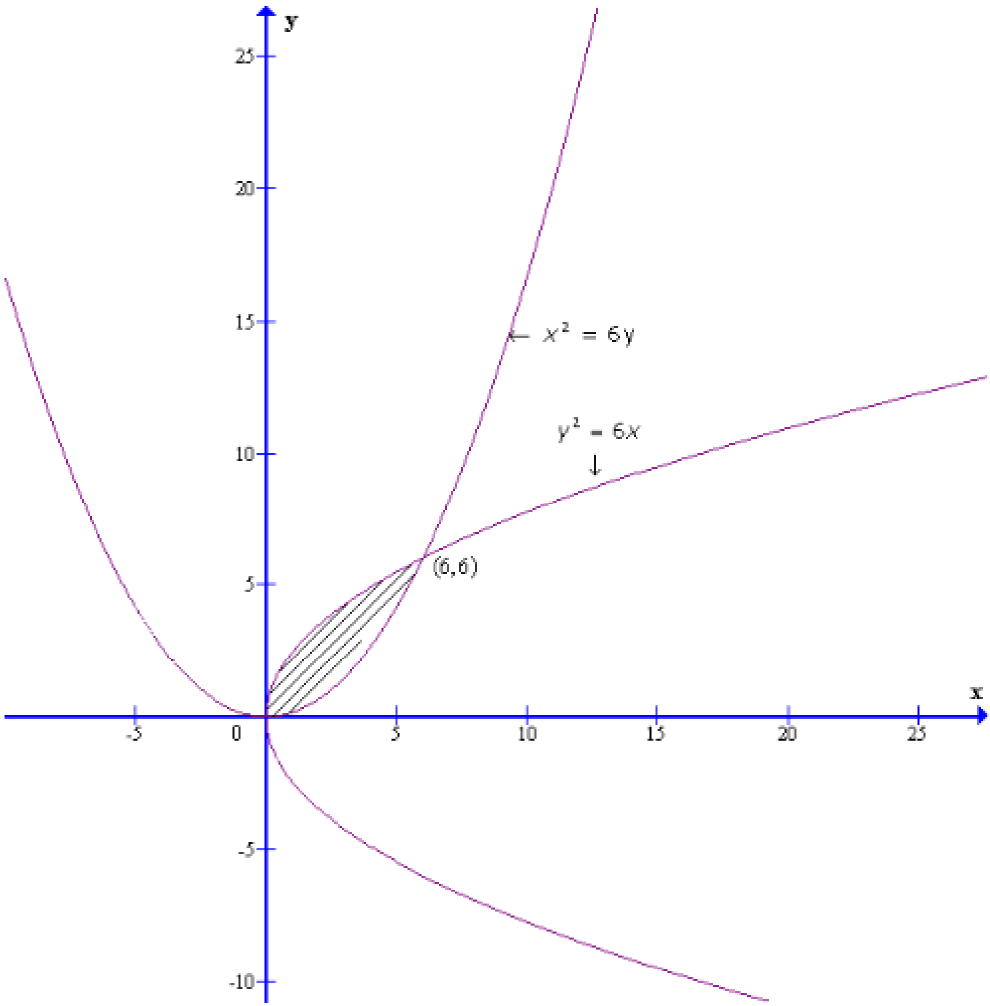
RD Sharma Solutions Class 12 Maths Chapter 21 Ex 21.3

Areas of Bounded Regions Ex-21-3 Q1



Area of the bounded region

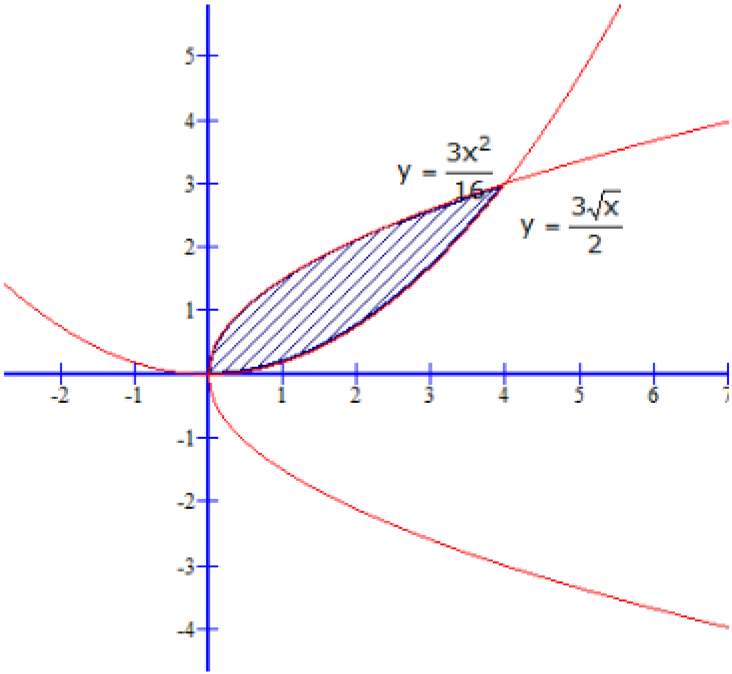
$$= \int_0^6 \sqrt{6x} - \frac{x^2}{6}\, dx$$

$$= \left[\sqrt{6}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{18}\right]_0^6$$

$$= \left[\sqrt{6}\,\frac{(6)^{3/2}}{3/2} - \frac{(6)^3}{18} - 0\right]$$

= 12 sq. units

Areas of Bounded Regions Ex-21-3 Q2



Area of the region $= \int_0^4 \left[\frac{3\sqrt{x}}{2} - \frac{3x^2}{16}\right] dx$

$= \left[x^{3/2} - \frac{x^3}{16}\right]_0^4$

$= \left[(4)^{3/2} - \frac{(4)^3}{16}\right]$

$= \left[8 - \frac{64}{16}\right]$

$= [8-4] = 4$ sq.units

Areas of Bounded Regions Ex-21-3 Q3

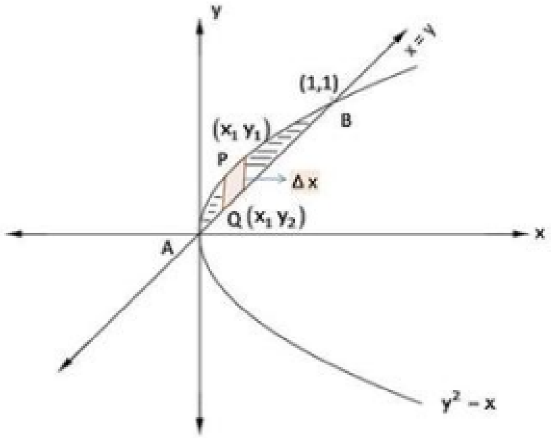
We have to find area of region bounded by

$y^2 = x$ ---(1)

and $y = x$ ---(2)

Equation (1) represents parabola with vertex (0,0) and axis as x-axis and equation (2) represents a line passing through origin and intersecting parabola at (0,0) and (1,1).

A rough sketch of curves is as below:-



Shaded region represents the required area. We slice it in rectangle with Width $= \Delta x$, length $= y_1 - y_2$

Area of rectangle $= (y_1 - y_2)\Delta x$

The approximation triangle can slide from $x = 0$ to $x = 1$.

Required area = region $AOBPA$

$$= \int_0^1 (y_1 - y_2)\,dx$$

$$= \int_0^1 \left(\sqrt{x} - x\right)dx$$

$$= \left[\frac{2}{3}x\sqrt{x} - \frac{x^2}{2}\right]_0^1$$

$$= \left[\frac{2}{3}.1.\sqrt{1} - \frac{(1)^2}{2}\right] - [0]$$

$$= \left[\frac{2}{3} - \frac{1}{2}\right]$$

Required area $= \frac{1}{6}$ square units

Areas of Bounded Regions Ex-21-3 Q4

We have to find area bounded by the curves

$$y = 4 - x^2$$

$$\Rightarrow \quad x^2 = -(y-4) \qquad ---(1)$$

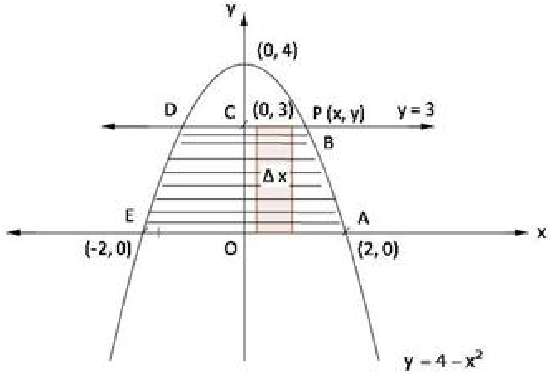
$$\text{and } y = 0 \qquad ---(2)$$

$$y = 3 \qquad ---(3)$$

Equation (1) represents a parabola with vertex (0,4) and passes through (0,2), (0,−2)

Equation (1) is x-axis and equation (3) is a line parallel to x-axis passing through (0,3).

A rough sketch of curves is below:-



Shaded region represents the required area. We slice it in approximation rectangle with its Width $= \Delta x$ and length $= y - 0 = y$

Area of the rectangle $= y\Delta x$.

This approximation rectangle can slide from $x = 0$ to $x = 2$ for region $OABCO$.

$$\text{Required area} = \text{Region } ABDEA$$

$$= 2(\text{Region } OABCO)$$

$$= 2\int_0^2 y\,dx$$

$$= 2\int_0^2 (4 - x^2)\,dx$$

$$= 2\left(4x - \frac{x^3}{3}\right)_0^2$$

$$= 2\left[\left(8 - \frac{8}{3}\right) - (0)\right]$$

Required area $= \frac{32}{3}$ square units

Areas of Bounded Regions Ex-21-3 Q5

Here to find area $\left\{(x,y): \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1 \le \frac{x}{a}+\frac{y}{b}\right\}$

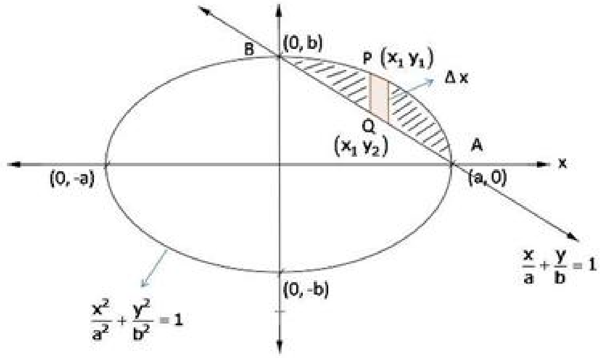
So,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad ---(1)$$

$$\frac{x}{a}+\frac{y}{b} = 1 \qquad ---(2)$$

Equation (1) represents ellipse with centre at origin and passing through $(\pm a,0)$, $(0,\pm b)$ equation (2) represents a line passing through $(a,0)$ and $(0,b)$.

A rough sketch of curves is below:- let $a > b$



Shaded region is the required region as by substituting $(0,0)$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1$ gives a true statement and by substituting $(0,0)$ in $1 \le \frac{x}{a}+\frac{y}{b}$ gives a false statement.

We slice the shaded region into approximation rectangles with Width $= \Delta x$, length $= (y_1 - y_2)$

Area of the rectangle $= (y_1 - y_2)$

The approximation rectangle can slide from $x = 0$ to $x = a$, so

$$\text{Required area} = \int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2} - \frac{b}{a}(a-x)\right]dx$$

$$= \frac{b}{a}\int_0^a \left[\sqrt{a^2-x^2} - (a-x)\right]dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)-ax+\frac{x^2}{2}\right]_0^a$$

$$= \frac{b}{a}\left[\left(\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}(1)-a^2+\frac{a^2}{2}\right)-(0+0+0+0)\right]$$

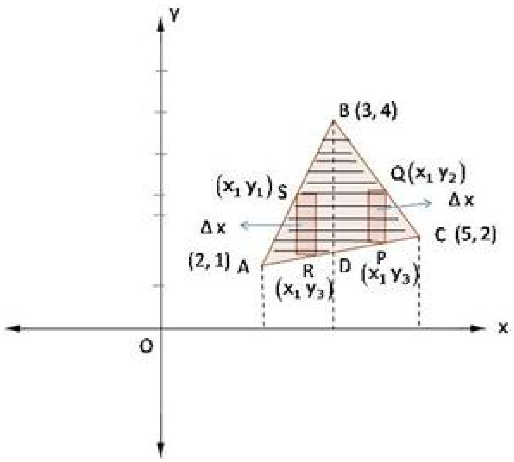
$$= \frac{b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}-\frac{a^2}{2}\right]$$

$$= \frac{b}{a}\frac{a^2}{2}\left(\frac{\pi-2}{2}\right)$$

Required area $= \frac{ab}{4}(\pi-2)$ square units

Areas of Bounded Regions Ex-21-3 Q6

Here we have find area of the triangle whose vertices are $A(2,1)$, $B(3,4)$ and $C(5,2)$



Equation of AB,

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 1 = \left(\frac{4-1}{3-2}\right)(x - 2)$$

$$y - 1 = \frac{3}{1}(x - 2)$$

$$y = 3x - 6 + 1$$

$$y = 3x - 5 \qquad \text{---}(1)$$

Equation of BC,

$$y - 4 = \left(\frac{2-4}{5-3}\right)(x - 3)$$

$$= \frac{-2}{2}(x - 3)$$

$$y - 4 = -x + 3$$

$$y = -x + 7 \qquad \text{---}(2)$$

Equation of AC,

$$y-1=\left(\frac{2-1}{5-2}\right)(x-2)$$

$$y-1=\frac{1}{3}(x-2)$$

$$y=\frac{1}{3}x-\frac{2}{3}+1$$

$$y=\frac{1}{3}x+\frac{1}{3} \qquad ---(3)$$

Shaded area $\triangle ABC$ is the required area.

$ar(\triangle ABC)=ar(\triangle ABD)+ar(\triangle BDC)$

For $ar(\triangle ABD)$: we slice the region into approximation rectangle with width $=\Delta x$ and length (y_1-y_3) area of rectangle $=(y_1-y_3)\Delta x$

This approximation rectangle slides from $x=2$ to $x=3$

$$ar(\triangle ABD)=\int_2^3 (y_1-y_3)dx$$

$$=\int_2^3\left[(3x-5)-\left(\frac{1}{3}x+\frac{1}{3}\right)\right]dx$$

$$=\int_2^3\left(3x-5-\frac{1}{3}x-\frac{1}{3}\right)dx$$

$$=\int_2^3\left(\frac{8x}{3}-\frac{16}{3}\right)dx$$

$$=\frac{8}{3}\left(\frac{x^2}{2}-12x\right)_2^3$$

$$=\frac{8}{3}\left[\left(\frac{9}{2}-6\right)-(2-4)\right]$$

$$=\frac{8}{3}\left[-\frac{3}{2}+2\right]$$

$$=\frac{8}{3}\times\frac{1}{2}$$

$$ar(\triangle ABD)=\frac{4}{3}\text{ sq. unit}$$

For $ar(\triangle BDC)$: we slice the region into rectangle with width $=\Delta x$ and length (y_2-y_3). Area of rectangle $=(y_2-y_3)\Delta x$

The approximation rectangle slides from $x = 3$ to $x = 5$.

$$\text{Area}\left(\triangle BDC\right) = \int_3^5 \left(y_2 - y_3\right) dx$$

$$= \int_3^5 \left[\left(-x + 7\right) - \left(\frac{1}{3}x + \frac{1}{3}\right)\right] dx$$

$$= \int_3^5 \left(-x + 7 - \frac{1}{3}x - \frac{1}{3}\right) dx$$

$$= \int_3^5 \left(-\frac{4}{3}x + \frac{20}{3}\right) dx$$

$$= -\left(\frac{4x^2}{6} - \frac{20}{3}x\right)_3^5$$

$$= -\left[\left(\frac{4(5)^2}{6} + \frac{20(5)}{3}\right) - \left(\frac{4(3)^2}{6} - \frac{20}{3}(3)\right)\right]$$

$$= -\left[\left(\frac{50}{3} - \frac{100}{3}\right) - \left(6 - 20\right)\right]$$

$$= -\left[-\frac{50}{3} + 14\right]$$

$$= -\left[-\frac{8}{3}\right]$$

$$ar\left(\triangle BDC\right) = \frac{8}{3} \text{ sq. units}$$

So, $ar\left(\triangle ABC\right) = ar\left(\triangle ABD\right) + ar\left(\triangle BDC\right)$

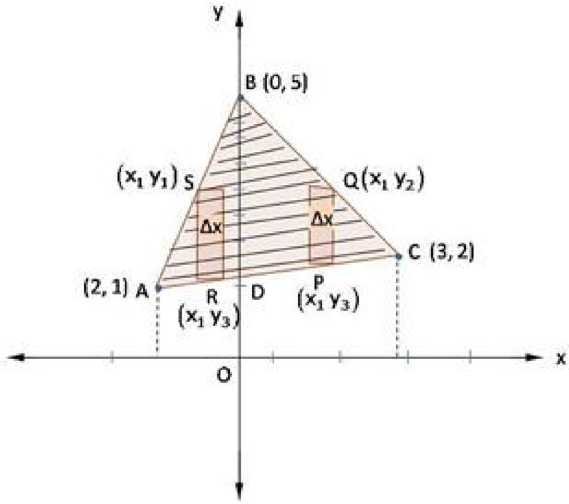
$$= \frac{4}{3} + \frac{8}{3}$$

$$= \frac{12}{3}$$

$$ar\left(\triangle ABC\right) = 4 \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q7

We have to find area of the triangle whose vertices are $A(-1,1)$, $B(0,5)$, $C(3,2)$



Equation of AB,

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 1 = \left(\frac{5-1}{0+1}\right)(x+1)$$

$$y - 1 = \frac{4}{1}(x+1)$$

$$y = 4x + 4 + 1$$

$$y = 4x + 5 \qquad ---(1)$$

Equation of BC,

$$y - 5 = \left(\frac{2-5}{3-0}\right)(x-0)$$

$$= \frac{-3}{3}(x-0)$$

$$y - 5 = -x$$

$$y = 5 - x \qquad ---(2)$$

Equation of AC,

$$y - 5 = \left(\frac{2-5}{3-0}\right)(x-0)$$

$$= \frac{-3}{3}(x-0)$$

$$y - 5 = -x$$

$$y = 5 - x \qquad ---(2)$$

Equation of AC,

$$y - 1 = \left(\frac{2-1}{3+1}\right)(x+1)$$

$$y - 1 = \frac{1}{4}(x+1)$$

$$y = \frac{1}{4}x + \frac{1}{4} + 1$$

$$y = \frac{1}{4}(x+5) \qquad ---(3)$$

Shaded area $\triangle ABC$ is the required area.

$ar(\triangle ABC) = ar(\triangle ABD) + ar(\triangle BDC)$

For $ar(\triangle ABD)$: we slice the region into approximation rectangle with width $= \Delta x$ and length $(y_1 - y_3)$ area of rectangle $= (y_1 - y_3)\Delta x$

This approximation rectangle slides from $x = -1$ to $x = 0$, so

$$ar(\triangle ABD) = \int_{-1}^{0}(y_1 - y_3)dx$$

$$= \int_{-1}^{0}\left[(4x+5) - \frac{1}{4}(x+5)\right]dx$$

$$= \int_{-1}^{0}\left(4x + 5 - \frac{x}{4} - \frac{5}{4}\right)dx$$

$$= \int_{-1}^{0}\left(\frac{15}{4}x + \frac{15}{4}\right)dx$$

$$= \frac{15}{4}\left(\frac{x^2}{2} + x\right)_{-1}^{0}$$

$$= \frac{15}{4}\left[(0) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{15}{4} \times \frac{1}{2}$$

$ar(\Delta ABD) = \frac{15}{8}$ sq. units

For $ar(\Delta BDC)$: we slice the region into rectangle with width $= \Delta x$ and length $(y_2 - y_3)$. Area of rectangle $= (y_2 - y_3)\Delta x$

The approximation rectangle slides from $x = 0$ to $x = 3$.

$$\text{Area}(\Delta BDC) = \int_0^3 (y_2 - y_3)dx$$

$$= \int_0^3 \left[(5 - x) - \left(\frac{1}{4}x + \frac{5}{4}\right)\right]dx$$

$$= \int_0^3 \left(5 - x - \frac{1}{4}x - \frac{5}{4}\right)dx$$

$$= \int_0^3 \left(-\frac{5}{4}x + \frac{15}{4}\right)dx$$

$$= \frac{5}{4}\left(3x - \frac{x^2}{2}\right)_0^3$$

$$= \frac{5}{4}\left[9 - \frac{9}{2}\right]$$

$ar(\Delta BDC) = \frac{45}{8}$ sq. units

So, $ar(\Delta ABC) = ar(\Delta ABD) + ar(\Delta BDC)$

$$= \frac{15}{8} + \frac{45}{8}$$

$$= \frac{60}{8}$$

$ar(\Delta ABC) = \frac{15}{2}$ sq. units

Areas of Bounded Regions Ex-21-3 Q8

To find area of triangular region bounded by

$y = 2x + 1$ (Say, line AB) ---(1)

$y = 3x + 1$ (Say, line BC) ---(2)

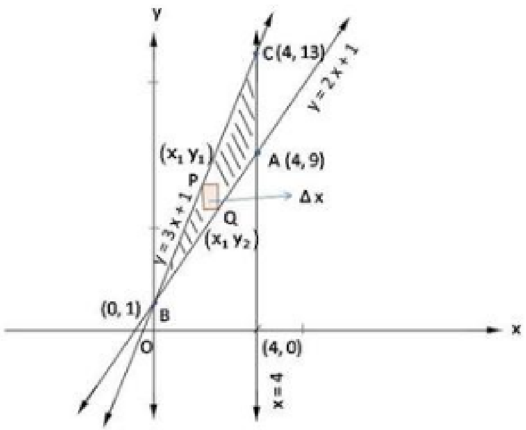
$y = 4$ (Say, line AC) ---(3)

equation (1) represents a line passing through points $(0,1)$ and $\left(-\frac{1}{2},0\right)$, equation (2) represents a line passing through points $(0,1)$ and $\left(-\frac{1}{3},0\right)$. Equation (3) represents a line parallel to y-axis passing through $(4,0)$.

Solving equation (1) and (2) gives point $B(0,1)$

Solving equation (2) and (3) gives point $C(4,13)$

Solving equation (1) and (3) gives point $A(4,9)$



Shaded region $ABCA$ gives required triangular region. We slice this region into approximation rectangle with width $= \Delta x$, length $= (y_1 - y_2)$.

Area of rectangle $= (y_1 - y_2)\Delta x$

This qpproximation rectangle slides from $x = 0$ to $x = 4$, so

Required area = (Region $ABCA$)

$= \int_0^4 (y_1 - y_2)dx$

$= \int_0^4 [(3x+1) - (2x+1)]dx$

$= \int_0^4 x\,dx$

$= \left[\frac{x^2}{2}\right]_0^4$

Required area = 8 sq. units

Areas of Bounded Regions Ex-21-3 Q9

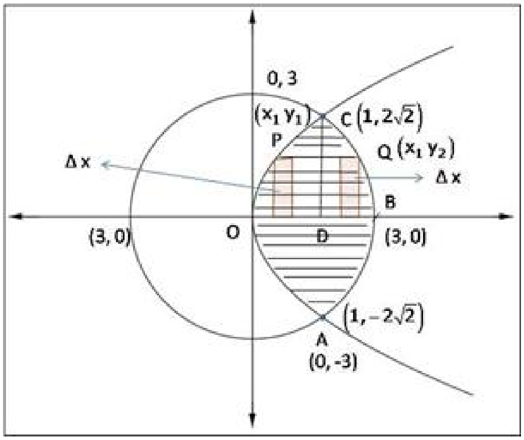
To find area $\{(x,y) : y^2 \le 8x, x^2 + y^2 \le 9\}$ given equation is

$y^2 = 8x$ --- (1)

$x^2 + y^2 = 9$ --- (2)

Equation (1) represents a parabola with vertex (0,0) and axis as x-axis, equation (2) represents a circle with centre (0,0) and radius $\sqrt{9} = 3$, so it meets area at $(\pm 3, 0)$, $(0, \pm 3)$. point of intersection of parabola and circle is $(1, 2\sqrt{2})$ and $(1, -2\sqrt{2})$.

A rough sketch of the curves is as below:-



Shaded region is the required region.

Required area = Region $OABCO$

$= 2(\text{Region } OBCO)$

Required area = $2(\text{region } ODCO + \text{region } DBCD)$

$$= 2\left[\int_0^1 \sqrt{8x}\,dx + \int_1^3 \sqrt{9 - x^2}\,dx\right]$$

$$= 2\left[\left(2\sqrt{2}.\frac{2}{3}x\sqrt{x}\right)_0^1 + \left(\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right)_1^3\right]$$

$$= 2\left[\left(\frac{4\sqrt{2}}{3}.1.\sqrt{1}\right) + \left\{\left(\frac{3}{2}.\sqrt{9-9} + \frac{9}{2}\sin^{-1}(1)\right) - \left(\frac{1}{2}\sqrt{9-1} + \frac{9}{2}\sin^{-1}\frac{1}{3}\right)\right\}\right]$$

$$= 2\left[\frac{4\sqrt{2}}{3} + \left\{\left(\frac{9}{2}.\frac{\pi}{2}\right) - \left(\frac{2\sqrt{2}}{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)\right\}\right]$$

$$= 2\left[\frac{4\sqrt{2}}{3} + \frac{9\pi}{4} - \sqrt{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

Required area $= 2\left[\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$ square units

Areas of Bounded Regions Ex-21-3 Q10

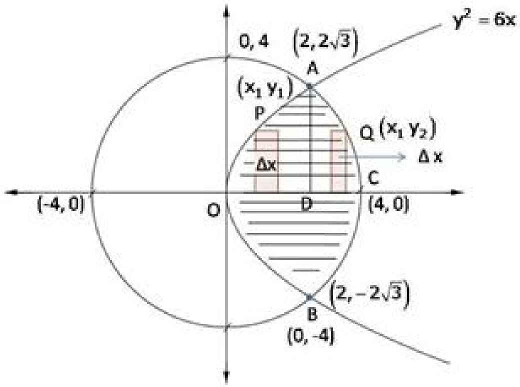
To find the area of common to

$x^2 + y^2 = 16$ --- (1)

$y^2 = 6x$ --- (2)

Equation (1) represents a parabola with vertex (0,0) and axis as x-axis, equation (2) represents a circle with centre (0,0) and radius $\sqrt{16} = 4$, so it meets areas at $(\pm 4,0)$, $(0,\pm 4,0)$. points of intersection of parabola and circle are $(2,2\sqrt{3})$ and $(2,-2\sqrt{3})$.

A rough sketch of the curves is as below:-



Shaded region represents the required area.

Required area = Region $OBCAO$

Required area = 2 (region $ODAO$ + region $DCAD$) --- (1)

Region $ODAO$ is divided into approximation rectangle with area $y_1 \Delta x$ and slides from $x = 0$ to $x = 2$. And region $DCAD$ is divided into approximation rectangle with area $y_2 \Delta x$ and slides from $x = 2$ and $x = 4$. So using equation (1),

Required area $= 2\left(\int_0^2 y_1 dx + \int_2^4 y_2 dx\right)$

$= 2\left[\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16-x^2}\,dx\right]$

$= 2\left[\left\{\sqrt{6}.\frac{2}{3}x\sqrt{x}\right\}_0^2 + \left\{\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right\}_2^4\right]$

$= 2\left[\left\{\sqrt{6}.\frac{2}{3}2.\sqrt{2}\right\} + \left\{\left(\frac{4}{2}\sqrt{16-16} + \frac{16}{2}\sin^{-1}\frac{4}{4}\right) - \left(\frac{2}{2}\sqrt{16-4} + \frac{16}{2}\sin^{-1}\frac{2}{4}\right)\right\}\right]$

$= 2\left[\frac{4}{3}\sqrt{12} + \left\{\left(0 + 8\sin^{-1}(1)\right) - \left(1.\sqrt{12} + 8\sin^{-1}\left(\frac{1}{2}\right)\right)\right\}\right]$

$= 2\left[\frac{8\sqrt{3}}{3} + \left\{\left(8.\frac{\pi}{2}\right) - \left(2\sqrt{3} + 8.\frac{\pi}{6}\right)\right\}\right]$

$= 2\left\{\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right\}$

$= 2\left\{\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right\}$

Required area $= \frac{4}{3}\left(4\pi + \sqrt{3}\right)$ sq.units

Areas of Bounded Regions Ex-21-3 Q11

Equation of the given circles are

$X^2 + y^2 = 4$ ...(1)

And $(x - 2)^2 + y^2 = 4$ ...(2)

Equation (1) is a circle with centre O at eh origin and radius 2. Equation (2) is a circle with centre C (2,0) and radius 2. Solving equations (1) and (2), we have

$(x - 2)^2 + y^2 = x^2 + y^2$

Or $x^2 - 4x + 4 + y^2 = x^2 + y^2$

Or $x = 1$ which gives $y \pm\sqrt{3}$

Thus, the points of intersection of the given circles are A $\left(1, \sqrt{3}\right)$ and A' $\left(1, -\sqrt{3}\right)$ as shown in the fig.,

Required area of the enclosed region OACA'O between circle

= 2 [area of the region ODCAO] (Why ?)

= 2 [area of the region ODAO + area of the region DCAD]

$= 2\left[\int_0^1 y\,dx + \int_1^2 y\,dx\right]$

$= 2\left[\int_0^1 \sqrt{4 - (x - 2)^2}\,dx + \int_1^2 \sqrt{4 - x^2}\,dx\right]$ (Why ?)

$= 2\left[\frac{1}{2}(x - 2)\sqrt{4 - (x - 2)^2} + \frac{1}{2} \times 4\sin^{-1}\left(\frac{x - 2}{2}\right)\right]_0^1 + 2\left[\frac{1}{2} \times \sqrt{4 - x^2} + \frac{1}{2} \times 4\sin^{-1}\frac{x}{2}\right]_1^2$

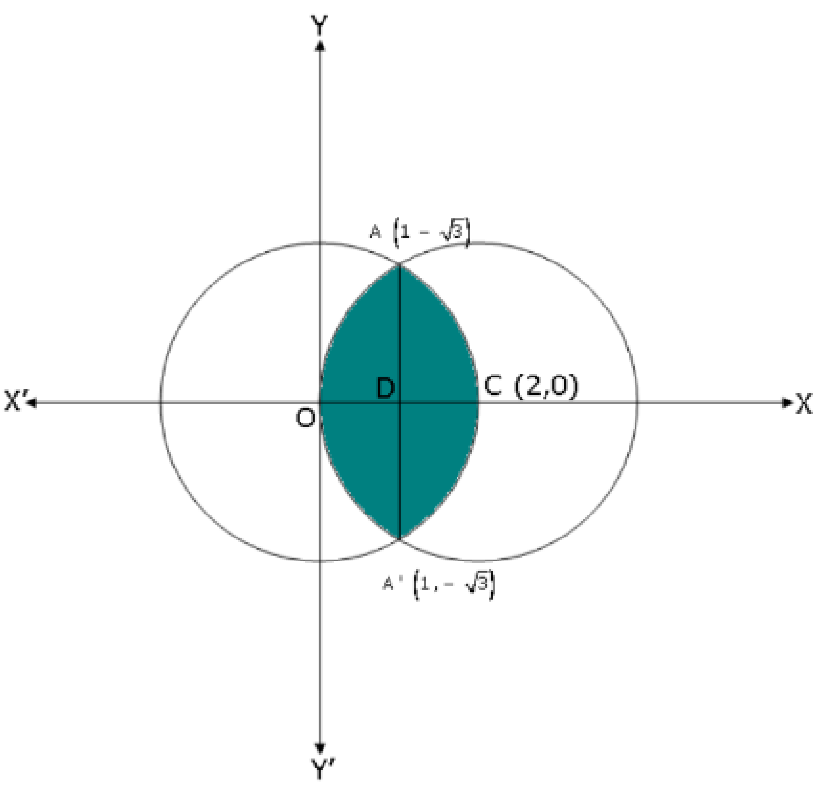
$= \left[(x - 2)\sqrt{4 - (x - 2)^2} + 4\sin^{-1}\left(\frac{x - 2}{2}\right)\right]_0^1 + \left[x\sqrt{4 - x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$

$= \left[\left(-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right) - 4\sin^{-1}(-1)\right] + \left[4\ \sin^{-1} 1 - \sqrt{3} - 4\ \sin^{-1}\frac{1}{2}\right]$

$= \left[\left(-\sqrt{3} - 4 \times \frac{\pi}{6}\right) + 4 \times \frac{\pi}{2}\right] + \left[4 \times \frac{\pi}{2} - \sqrt{3} - 4 \times \frac{\pi}{6}\right]$

$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3}\right)$

$= \frac{8\pi}{3} - 2\sqrt{3}$ square units



Areas of Bounded Regions Ex-21-3 Q12

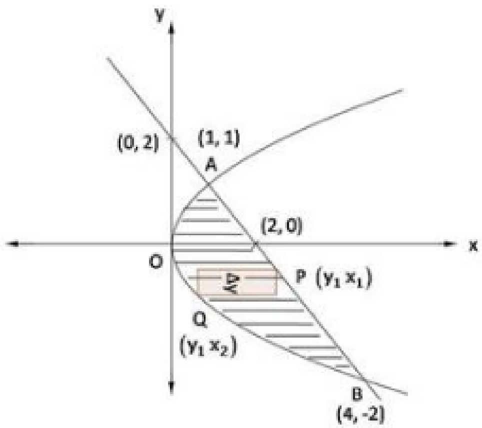
To find region enclosed by

$y^2 = x$ ---(1)

$x + y = 2$ ---(2)

Equation (1) represents a parabola with vertex at origin and its axis as x-axis, equation (2) represents a line passing through (2,0) and (0,2). points of intersection of line and parabola are (1,1) and (4,-2).

A rough sketch of curves is as below:-



Shaded region represents the required area. We slice it in rectangles of width Δy and length = $(x_1 - x_2)$.

Area of rectangle = $(x_1 - x_2)\Delta y$.

This approximation rectangle slides from $y = -2$ to $y = 1$, so

Required area = Region $AOBA$

$$= \int_{-2}^{1}(x_1 - x_2)\,dy$$

$$= \int_{-2}^{1}(2 - y - y^2)\,dy$$

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

$$= \left[\left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right)\right]$$

$$= \left[\left(\frac{12 - 3 - 2}{6}\right) - \left(\frac{-12 - 6 + 8}{3}\right)\right]$$

$$= \frac{7}{6} + \frac{10}{3}$$

Required area = $\frac{9}{2}$ sq.units

Areas of Bounded Regions Ex-21-3 Q13

To find area $\left\{(x,y) : y^2 \le 3x, 3x^2 + 3y^2 \le 16\right\}$

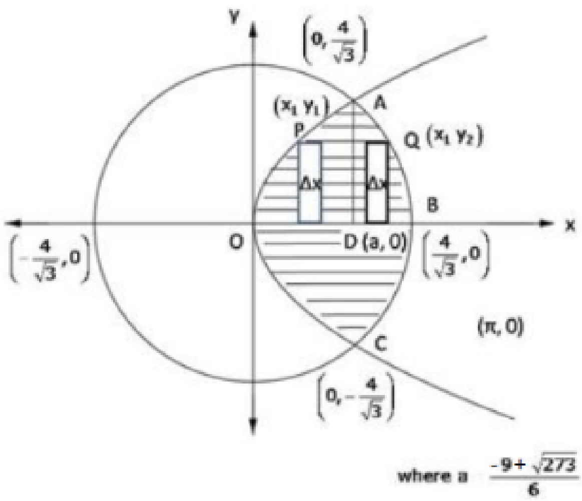
$$\Rightarrow \quad y^2 = 3x \qquad ---(1)$$

$$3x^2 + 3y^2 = 16$$

$$x^2 + y^2 = \frac{16}{3} \qquad ---(2)$$

Equation (1) represents a parabola with vertex $(0,0)$ and axis as x-axis,

equation (2) represents a circle with centre $(0,0)$ and radius $\frac{4}{\sqrt{3}}$ and meets axes at $\left(\pm\frac{4}{\sqrt{3}},0\right)$ and $\left(0,\pm\frac{4}{\sqrt{3}}\right)$. A rough sketch of the curves is given below:-



Required area = Region $OCBAO$

= 2 (Region $OBAO$)

= 2 (Region $ODAO$ + Region $DBAD$)

$$= 2\left[\int_0^a \sqrt{3x}\,dx + \int_a^{\frac{4}{\sqrt3}} \sqrt{\left(\frac{4}{\sqrt3}\right)^2 - x^2}\,dx\right]$$

$$A = 2\left[\left(\sqrt{3}.\frac{2}{3}x\sqrt{x}\right)_0^a + \left(\frac{x}{2}\sqrt{\left(\frac{4}{\sqrt3}\right)^2 - x^2} + \frac{16}{6}\sin^{-1}\frac{x\sqrt3}{4}\right)_a^{\frac{4}{\sqrt3}}\right]$$

$$= 2\left[\left(\frac{2}{\sqrt3}a\sqrt{a}\right) + \left\{\left(0 + \frac{8}{3}\sin^{-1}(1)\right) - \left(\frac{a}{2}\sqrt{\left(\frac{4}{\sqrt3}\right)^2 - a^2} + \frac{8}{3}\sin^{-1}\frac{a\sqrt3}{4}\right)\right\}\right]$$

Thus, $A = \frac{4}{\sqrt3}a^{\frac{3}{2}} + \frac{8\pi}{3} - a\sqrt{\frac{16}{3} - a^2} - \frac{16}{3}\sin^{-1}\left(\frac{\sqrt3 a}{4}\right)$

Where, $a = \frac{-9+\sqrt{273}}{6}$

Areas of Bounded Regions Ex-21-3 Q14

To find area $\{(x,y) : y^2 \le 5x, 5x^2 + 5y^2 \le 36\}$

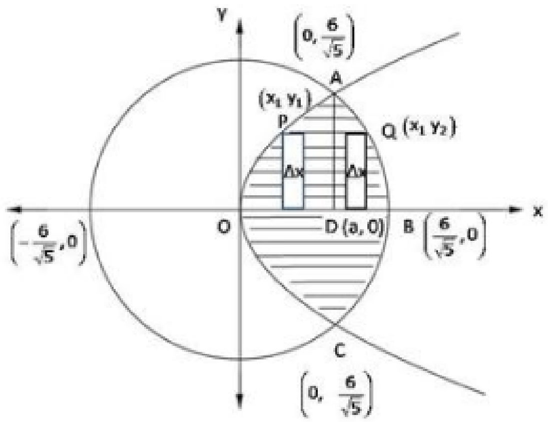
$$\Rightarrow \quad y^2 = 5x \qquad ---(1)$$

$$5x^2 + 5y^2 = 36$$

$$x^2 + y^2 = \frac{36}{5} \qquad ---(2)$$

Equation (1) represents a parabola with vertex (0,0) and axis as x-axis.

Equation (2) represents a circle with centre (0,0) and radius $\frac{6}{\sqrt{5}}$ and meets axes at $\left(\pm\frac{6}{\sqrt{5}},0\right)$ and $\left(0,\pm\frac{6}{\sqrt{5}}\right)$. x ordinate of point of intersection of circle and parabola is a where $a = \frac{-25+\sqrt{1345}}{10}$. A rough sketch of curves is:-



Required area = Region $OCBAO$

$$A = 2(\text{Region } OBAO)$$

$$= 2(\text{Region } ODAO + \text{Region } DBAD)$$

$$= 2\left[\int_0^a \sqrt{5x}\,dx + \int_a^{\frac{6}{\sqrt{5}}} \sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 - x^2}\,dx\right]$$

$$= 2\left[\left(\sqrt{5}.\frac{2}{3}x\sqrt{x}\right)_0^a + \left(\frac{x}{2}\sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 - x^2} + \frac{36}{10}\sin^{-1}\left(\frac{x\sqrt{5}}{6}\right)\right)_a^{\frac{6}{\sqrt{5}}}\right]$$

$$= \frac{4\sqrt{5}}{3}a\sqrt{a} + 2\left\{\left(0 + \frac{18}{5}.\frac{\pi}{2}\right) - \left(\frac{a}{2}\sqrt{\left(\frac{6}{\sqrt{5}}\right)^2 - a^2} + \frac{18}{5}\sin^{-1}\left(\frac{a\sqrt{5}}{6}\right)\right)\right\}$$

Thus, $A = \frac{4\sqrt{5}}{a}a^{\frac{3}{2}} + \frac{18\pi}{5} - a\sqrt{\frac{36}{5} - a^2} - \frac{36}{5}\sin^{-1}\left(\frac{a\sqrt{5}}{6}\right)$

Where, $a = \frac{-25+\sqrt{1345}}{10}$

Areas of Bounded Regions Ex-21-3 Q15

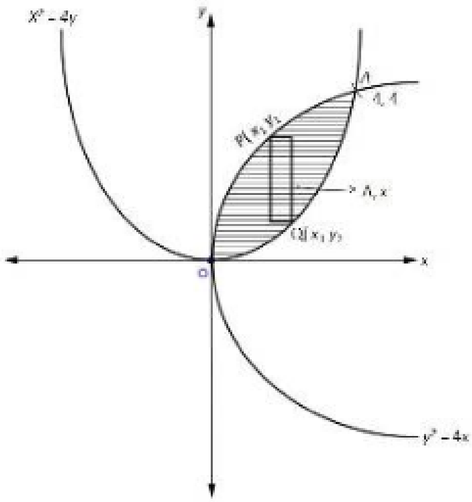
To find area bounded by

$y^2 = 4x$ ---(1)

$x^2 = 4y$ ---(2)

Equation (1) represents a parabola with vertex (0,0) and axis as x-axis.
Equation (2) represents a parabola with vertex (0,0) and axis as y-axis.
Points of intersection of parabolas are (0,0) and (4,4).

A rough sketch is given as:-



The shaded region is required area and it is sliced into rectangles with width Δx and length $(y_1 - y_2)$. Area of rectangle = $(y_1 - y_2)\Delta x$.

This approximation rectangle slide from $x = 0$ to $x = 4$, so

Required area = Region $OQAPO$

$$A = \int_0^4 (y_1 - y_2)dx$$
$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx$$
$$= \left[2.\frac{2}{3}x\sqrt{x} - \frac{x^3}{12}\right]_0^4$$
$$= \left[\left(\frac{4}{3}.4\sqrt{4} - \frac{64}{12}\right) - (0)\right]$$

$$A = \frac{32}{3} - \frac{16}{3}$$

$$A = \frac{16}{3} \text{ sq.units}$$

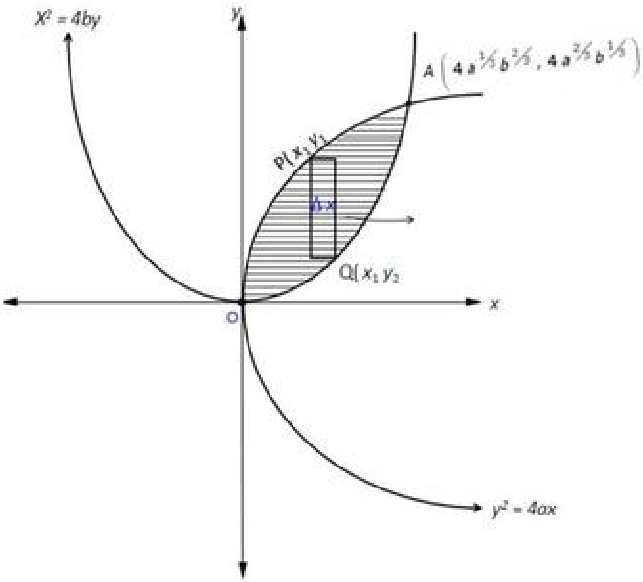
Areas of Bounded Regions Ex-21-3 Q16

To find area enclosed by

$$y^2 = 4ax \qquad ---(1)$$
$$x^2 = 4by \qquad ---(2)$$

Equation (1) represents a parabola with vertex $(0,0)$ and axis as x-axis,
equation (2) represents a parabola with vertex $(0,0)$ and axis as y-axis,
points of intersection of parabolas are $(0,0)$ and $\left(4a^{\frac{1}{3}} b^{\frac{2}{3}}, 4a^{\frac{2}{3}} b^{\frac{1}{3}}\right)$

A rough sketch is given as:-



The shaded region is required area and it is sliced into rectangles of width = Δx and length $(y_1 - y_2)$.

Area of rectangle = $(y_1 - y_2)\Delta x$.

This approximation rectangle slides from $x = 0$ to $x = 4a^{\frac{1}{3}} b^{\frac{2}{3}}$, so

Required area = Region $OQAPO$

$$= \int_0^{4a^{\frac{1}{3}} b^{\frac{2}{3}}} \left(y_1 - y_2\right) dx$$

$$= \int_0^{4a^{\frac{1}{3}} b^{\frac{2}{3}}} \left(2\sqrt{a}.\sqrt{x} - \frac{x^2}{4b}\right) dx$$

$$= \left[2\sqrt{a}.\frac{2}{3}x\sqrt{x} - \frac{x^3}{12b}\right]_0^{4a^{\frac{1}{3}} b^{\frac{2}{3}}}$$

$$= \frac{32\sqrt{a}}{3}.a^{\frac{1}{3}} b^{\frac{2}{3}} a^{\frac{1}{6}} b^{\frac{1}{3}} - \frac{64ab^2}{12b}$$

$$= \frac{32}{3}ab - \frac{16}{3}ab$$

$A = \frac{16}{3}ab$ sq.units

Areas of Bounded Regions Ex-21-3 Q17

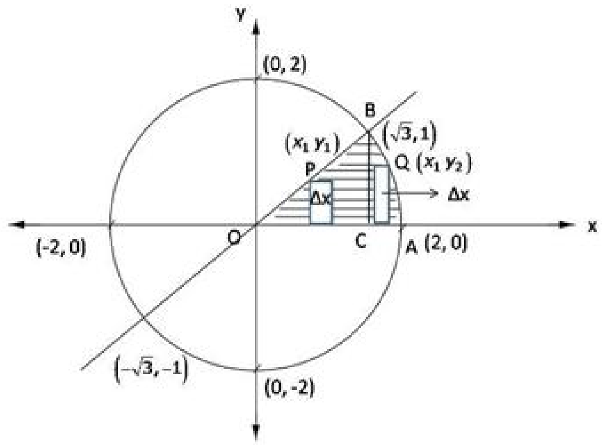
To find area in first quadrant enclosed by x-axis.

$x = \sqrt{3}y$ --- (1)

$x^2 + y^2 = 4$ --- (2)

Equation (1) represents a line passing through $(0,0), (-\sqrt{3},-1), (\sqrt{3},1)$. Equation (2) represents a circle with centre $(0,0)$ and passing through $(\pm 2,0), (0,\pm 2)$. Points of intersection of line and circle are $(-\sqrt{3},-1)$ and $(\sqrt{3},1)$.

A rough sketch of curves is given below:-



Required area = Region $OABO$

A = Region $OCBO$ + Region $ABCA$

$= \int_0^{\sqrt{3}} y_1 dx + \int_{\sqrt{3}}^2 y_2 dx$

$= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}} dx + \int_{\sqrt{3}}^2 \sqrt{4 - x^2} dx$

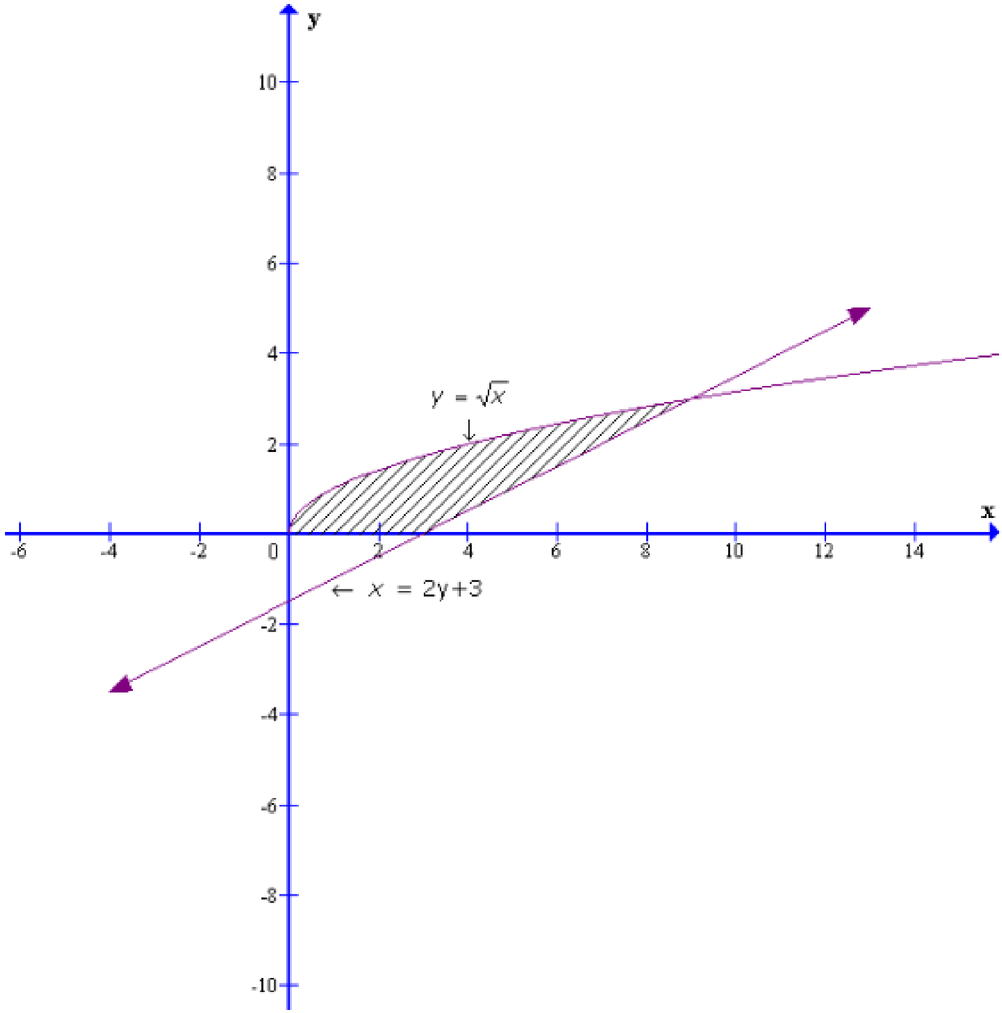
$= \left(\frac{x^2}{2\sqrt{3}}\right)_0^{\sqrt{3}} + \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^2$

$= \left(\frac{3}{2\sqrt{3}} - 0\right) + \left[\left(0 + 2\sin^{-1}(1)\right) - \left(\frac{\sqrt{3}}{2}.1 + 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$

$= \frac{\sqrt{3}}{2} + 2.\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2.\frac{\pi}{3}$

$A = \frac{\pi}{3}$ sq.units

Areas of Bounded Regions Ex-21-3 Q18



Area of the bounded region

$$= \int_0^3 \sqrt{x}\, dx + \int_3^9 \sqrt{x} - \left(\frac{x-3}{2}\right) dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_0^3 + \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{4} + \frac{3x}{2}\right]_3^9$$

$$= \left[\frac{(3)^{3/2}}{3/2} - 0\right] + \left[\frac{(9)^{3/2}}{3/2} - \frac{(9)^2}{4} + \frac{3(9)}{2} - \frac{(3)^{3/2}}{3/2} + \frac{(3)^2}{4} - \frac{3(3)}{2}\right]$$

= 9 sq. units

Areas of Bounded Regions Ex-21-3 Q19

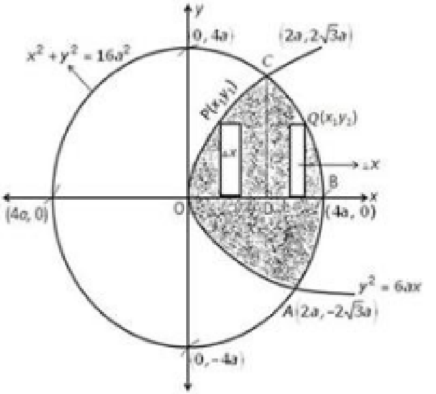
To find area in enclosed by

$x^2 + y^2 = 16a^2$ ---(1)

and $y^2 = 6ax$ ---(2)

Equation (1) represents a circle with centre $(0,0)$ and meets axes $(\pm 4a, 0), (0, \pm 4a)$.
Equation (2) represents a parabola with vertex $(0,0)$ and axis as x-axis. Points of intersection of circle and parabola are $(2a, 2\sqrt{3}a), (2a, -2\sqrt{3}a)$.

A rough sketch of curves is given as:-



Region $ODCO$ is sliced into rectangles of area $= y_1 \Delta x$ and it slides from $x = 0$ to $x = 2a$.

Region $BCDB$ is sliced into rectangles of area $= y_2 \Delta x$ it slides from $x = 2a$ to $x = 4a$. So,

Required area $= 2[\text{Region } ODCO + \text{Region } BCDB]$

$$= 2\left[\int_0^{2a} y_1 dx + \int_{2a}^{4a} y_2 dx\right]$$

$$= 2\left[\int_0^{2a} \sqrt{6ax}\, dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2}\, dx\right]$$

$$= 2\left[\sqrt{6a}.\left(\frac{2}{3} x\sqrt{x}\right)_0^{2a} + \left[\frac{x}{2}\sqrt{16a^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\left(\frac{x}{4a}\right)\right]_{2a}^{4a}\right]$$

$$= 2\left[\left(\sqrt{6a}.\frac{2}{3} 2a\sqrt{2a}\right) + \left[\left(0 + 8a^2.\frac{\pi}{2}\right) - \left(a\sqrt{12a^2} + 8a^2.\frac{\pi}{6}\right)\right]\right]$$

$$= 2\left[\frac{8\sqrt{3}a^2}{3} + 4a^2\pi - 2\sqrt{3}a^2 - \frac{4}{3}a^2\pi\right]$$

$$= 2\left[\frac{2\sqrt{3}a^2}{3} + \frac{8a^2\pi}{3}\right]$$

$$A = \frac{4a^2}{3}\left(4\pi + \sqrt{3}\right) \text{ sq.units}$$

Areas of Bounded Regions Ex-21-3 Q20

To find area lying above x-axis and included in the circle

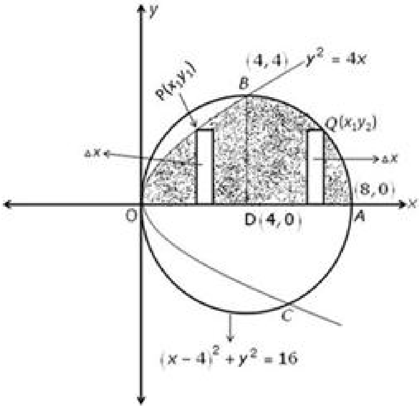
$x^2 + y^2 = 8x$

$(x-4)^2 + y^2 = 16$ ---(1)

and $y^2 = 4x$ ---(2)

Equation (1) represents a circle with centre $(4,0)$ and meets axes at $(0,0)$ and $(8,0)$.
Equation (2) represent a parabola with vertex $(0,0)$ and axis as x-axis. They intersect at $(4,-4)$ and $(4,4)$.

A rough sketch of the curves is as under:-



Shaded region is the required region

Required area = Region $OABO$

Required area = Region $ODBO$ + Region $DABD$ ---(1)

Region $ODBO$ is sliced into rectangles of area $y_1 \Delta x$. This approximation rectangle can slide from $x = 0$ to $x = 4$. So,

$$\text{Region } ODBO = \int_0^4 y_1 dx$$

$$= \int_0^4 2\sqrt{x}\,dx$$

$$= 2\left(\frac{2}{3}x\sqrt{x}\right)_0^4$$

Region $ODBO = \frac{32}{3}$ sq. units $\quad ---(2)$

Region $DABD$ is sliced into rectangles of area $y_2 \Delta x$. Which moves from $x = 4$ to $x = 8$. So,

$$\text{Region } DABD = \int_4^8 y_2 dx$$
$$= \int_4^8 \sqrt{16 - (x - 4)^2} dx$$
$$= \left[\frac{(x-4)}{2}\sqrt{16-(x-4)^2} + \frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_4^8$$
$$= \left[\left(0 + 8.\frac{\pi}{2}\right) - (0+0)\right]$$

Region $DABD = 4\pi$ sq. units $\quad ---(3)$

Using $(1),(2)$ and (3), we get

$$\text{Required area} = \left(\frac{32}{3} + 4\pi\right)$$
$$A = 4\left(\pi + \frac{8}{3}\right) \text{ sq.units}$$

Areas of Bounded Regions Ex-21-3 Q21

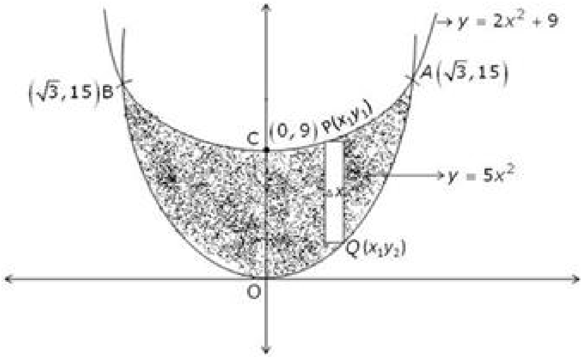
To find area enclosed by

$y = 5x^2$ ---(1)

$y = 2x^2 + 9$ ---(2)

Equation (1) represents a parabola with vertex (0,0) and axis as y-axis. Equation (2) represents a parabola with vertex (0,9) and axis as y-axis. Points of intersection of parabolas are $(\sqrt{3}, 15)$ and $(-\sqrt{3}, 15)$.

A rough sketch of curves is given as:-



Region $AOCA$ is sliced into rectangles with area $(y_1 - y_2)\Delta x$. It slides from $x = 0$ to $x = \sqrt{3}$, so

Required area = Region $AOBCA$

$= 2(\text{Region } AOCA)$

$= 2\int_0^{\sqrt{3}} (y_1 - y_2)\,dx$

$= 2\int_0^{\sqrt{3}} (2x^2 + 9 - 5x^2)\,dx$

$= 2\int_0^{\sqrt{3}} (9 - 3x^2)\,dx$

$= 2\left[9x - x^3\right]_0^{\sqrt{3}}$

$= 2\left[(9\sqrt{3} - 3\sqrt{3}) - (0)\right]$

Required area $= 12\sqrt{3}$ sq.units

Areas of Bounded Regions Ex-21-3 Q22

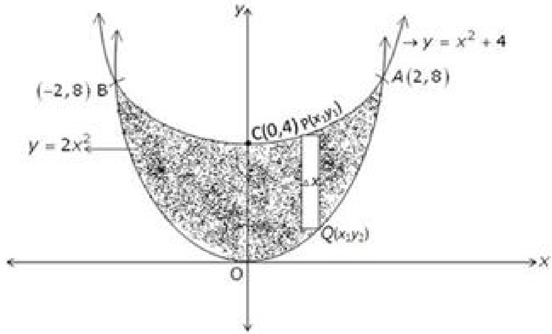
To find area enclosed by

$y = 2x^2$ --- (1)

$y = x^2 + 4$ --- (2)

Equation (1) represents a parabola with vertex (0,0) and axis as y-axis. Equation (2) represents a parabola with vertex (0,4) and axis as y-axis. Points of intersection of parabolas are (2,8) and (−2,8).

A rough sketch of curves is given as:-



Region $AOCA$ is sliced into rectangles with area $(y_1 - y_2)\Delta x$. And it slides from $x = 0$ to $x = 2$

Required area = Region $AOBCA$

$A = 2(\text{Region } AOCA)$

$= 2\int_0^2 (y_1 - y_2)\,dx$

$= 2\int_0^2 \left(x^2 + 4 - 2x^2\right)dx$

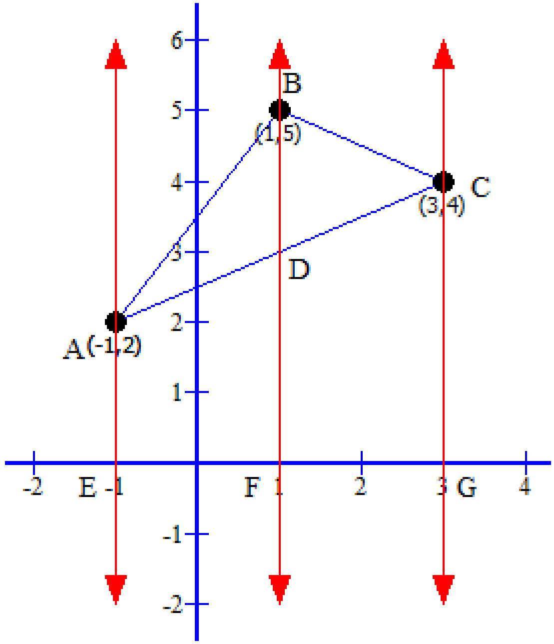
$= 2\int_0^2 \left(4 - x^2\right)dx$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 2\left[\left(8 - \frac{8}{3}\right) - (0)\right]$$

$A = \frac{32}{3}$ sq.units

Areas of Bounded Regions Ex-21-3 Q23



Equation of side AB,

$$\frac{x+1}{1+1} = \frac{y-2}{5-2}$$

$$\Rightarrow \frac{x+1}{2} = \frac{y-2}{3}$$

$$\Rightarrow 3x + 3 = 2y - 4$$

$$\Rightarrow 2y - 3x = 7$$

$$\therefore\ y = \frac{3x+7}{2} \ldots\ldots.(i)$$

Equation of side BC,

$$\frac{x-1}{3-1} = \frac{y-5}{4-5}$$

$$\Rightarrow \frac{x-1}{2} = \frac{y-5}{-1}$$

$$\Rightarrow -x + 1 = 2y - 10$$

$$\Rightarrow 2y = 11 - x$$

$$\therefore\ y = \frac{11-x}{2} \ldots\ldots(ii)$$

Equation of side AC,

$$\frac{x+1}{3+1}=\frac{y-2}{4-2}$$

$$\Rightarrow \frac{x+1}{4}=\frac{y-2}{2}$$

$$\Rightarrow \frac{x+1}{2}=\frac{y-2}{1}$$

$$\Rightarrow x+1=2y-4$$

$$\Rightarrow 2y=5+x$$

$$\therefore y=\frac{5+x}{2}$$

Area of required region

= Area of EABFE + Area of BFGCB - Area of AEGCA

$$=\int_{-1}^{1} y_{AB}dx+\int_{1}^{3} y_{BC}dx-\int_{-1}^{3} y_{AC}dx$$

$$=\int_{-1}^{1}\frac{3x+7}{2}dx+\int_{1}^{3}\frac{11-x}{2}dx-\int_{-1}^{3}\frac{5+x}{2}dx$$

$$=\frac{1}{2}\left[\frac{3x^2}{2}+7x\right]_{-1}^{1}+\frac{1}{2}\left[11x-\frac{x^2}{2}\right]_{1}^{3}-\frac{1}{2}\left[5x+\frac{x^2}{2}\right]_{-1}^{3}$$

$$=\frac{1}{2}\left[\frac{3(1^2-1^2)}{2}+7(1-(-1))\right]+\frac{1}{2}\left[11(3-1)-\frac{(3)^2-1^2}{2}\right]$$

$$-\frac{1}{2}\left[5(3-(-1))+\frac{(3)^2-1^2}{2}\right]$$

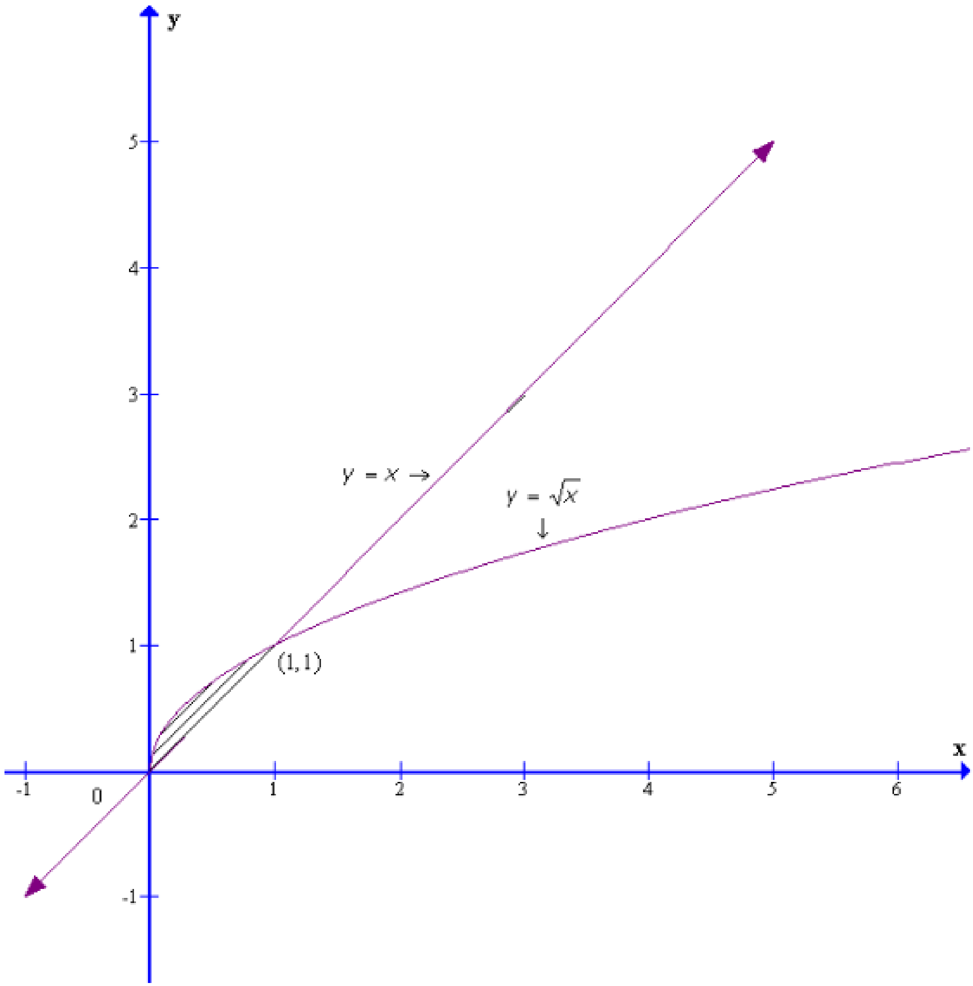
$$=\frac{1}{2}[0+14]+\frac{1}{2}[22-4]-\frac{1}{2}[20+4]$$

$$=7+\frac{1}{2}\times 18-\frac{1}{2}\times 24$$

$$=7+9-12$$

$$=4 \text{ sq units}$$

Areas of Bounded Regions Ex-21-3 Q24



Area of the bounded region

$= \int_0^1 \sqrt{x} - x \, dx$

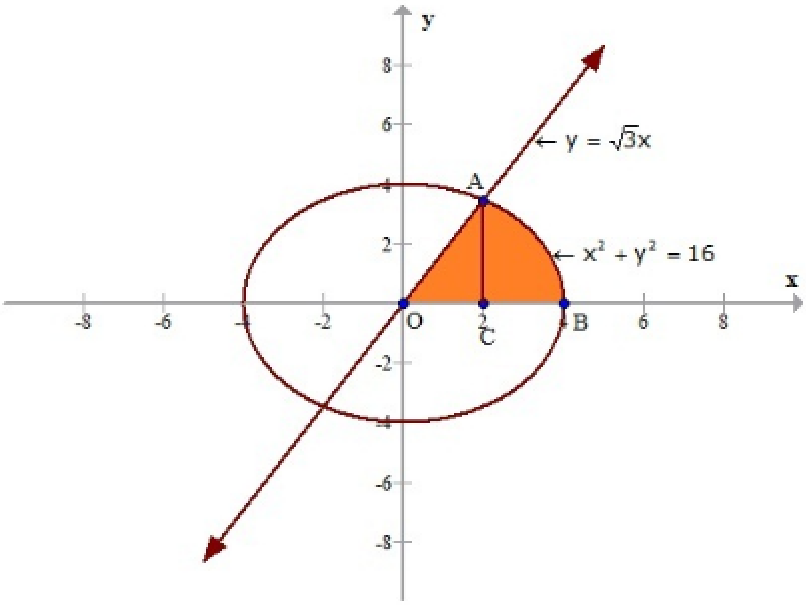
$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^1$

$= \left[\frac{2}{3} - \frac{1}{2} \right]$

$= \frac{1}{6}$ sq. units

Areas of Bounded Regions Ex-21-3 Q25

Consider the following graph.



We have, $y = \sqrt{3}x$

Substituting this value in $x^2 + y^2 = 16$,

$x^2 + (\sqrt{3}x)^2 = 16$

$\Rightarrow x^2 + 3x^2 = 16$

$\Rightarrow 4x^2 = 16$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

Since the shaded region is in the first quadrant, let us take the positive value of x.

Therefore, $x = 2$ *and* $y = 2\sqrt{3}$ *are the coordinates of the intersection point A.*

Thus, area of the shaded region OAB = Area OAC + Area ACB

$$\Rightarrow \text{Area } OAB = \int_0^2 \sqrt{3}x\,dx + \int_2^4 \sqrt{16 - x^2}\,dx$$

$$\Rightarrow \text{Area } OAB = \left(\frac{\sqrt{3}x^2}{2}\right)_0^2 + \frac{1}{2}\left[x\sqrt{16 - x^2} + 16\sin^{-1}\left(\frac{x}{4}\right)\right]_2^4$$

$$\Rightarrow \text{Area } OAB = \left(\frac{\sqrt{3} \times 4}{2}\right) + \frac{1}{2}\left[16\sin^{-1}\left(\frac{4}{4}\right)\right] - \frac{1}{2}\left[4\sqrt{16 - 12} + 16\sin^{-1}\left(\frac{2}{4}\right)\right]$$

$$\Rightarrow \text{Area } OAB = 2\sqrt{3} + \frac{1}{2}\left[16 \times \frac{\pi}{2}\right] - \frac{1}{2}\left[4\sqrt{3} + 16\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$\Rightarrow \text{Area } OAB = 2\sqrt{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}$$

$$\Rightarrow \text{Area } OAB = 4\pi - \frac{4\pi}{3}$$

$$\Rightarrow \text{Area } OAB = \frac{8\pi}{3} \text{ sq. units.}$$

Areas of Bounded Regions Ex-21-3 Q26

To find area bounded by

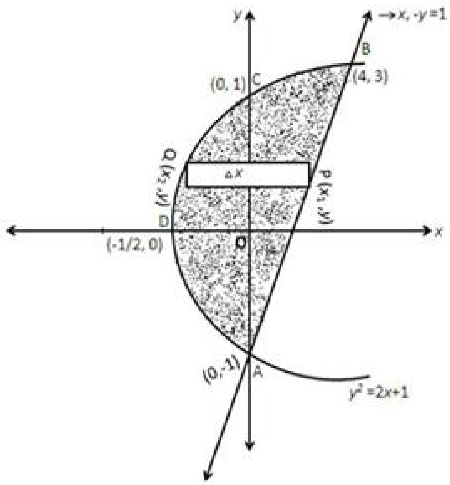
$y^2 = 2x + 1$ ---(1)

and $x - y = 1$ ---(2)

Equation (1) is a parabola with vertex $\left(-\frac{1}{2}, 0\right)$ and passes through $(0,1), (0,-1)$.

Equation (2) is a line passing through $(1,0)$ and $(0,-1)$. Points of intersection of parabola and line are $(3,2)$ and $(0,-1)$.

A rough sketch of the curves is given as:-



Shaded region represents the required area. It is sliced in rectangles of area $(x_1 - x_2)\Delta y$.
It slides from $y = -1$ to $y = 3$, so

Required area = Region $ABCDA$

$$= \int_{-1}^{3}(x_1 - x_2)dy$$

$$= \int_{-1}^{3}\left(1 + y - \frac{y^2 - 1}{2}\right)dy$$

$$= \frac{1}{2}\int_{-1}^{3}\left(2 + 2y - y^2 + 1\right)dy$$

$$= \frac{1}{2}\int_{-1}^{3}\left(3 + 2y - y^2\right)dy$$

$$= \frac{1}{2}\left[3y + y^2 - \frac{y^3}{3}\right]_{-1}^{3}$$

$$= \frac{1}{2}\left[(9 + 9 - 9) - \left(-3 + 1 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[9 + \frac{5}{3}\right]$$

$$= \frac{32}{6}$$

Required area $= \frac{16}{3}$ sq. units

Areas of Bounded Regions Ex-21-3 Q27

To find region bounded by curves

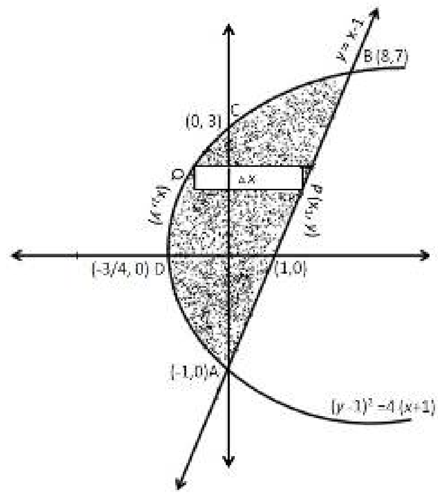
$y = x - 1$ ---(1)

and $(y-1)^2 = 4(x+1)$ ---(2)

Equation (1) represents a line passing through $(1,0)$ and $(0,-1)$ equation (2) represents a parabola with vertex $(-1,1)$ passes through $(0,3), (0,-1), \left(-\frac{3}{4}, 0\right)$.

Their points of intersection $(0,-1)$ and $(8,7)$.

A rough sketch of curves is given as:-



Shaded region is required area. It is sliced in rectangles of area $(x_1 - x_2)\Delta y$.
It slides from $y = -1$ to $y = 7$, so

Required area = Region $ABCDA$

$$A = \int_{-1}^{7} (x_1 - x_2)\, dy$$

$$= \int_{-1}^{7} \left(y + 1 - \frac{(y-1)^2}{4} + 1 \right) dy$$

$$= \frac{1}{4} \int_{-1}^{7} \left(4y + 4 - y^2 - 1 + 2y + 4 \right) dy$$

$$= \frac{1}{4} \int_{-1}^{7} \left(6y + 7 - y^2 \right) dy$$

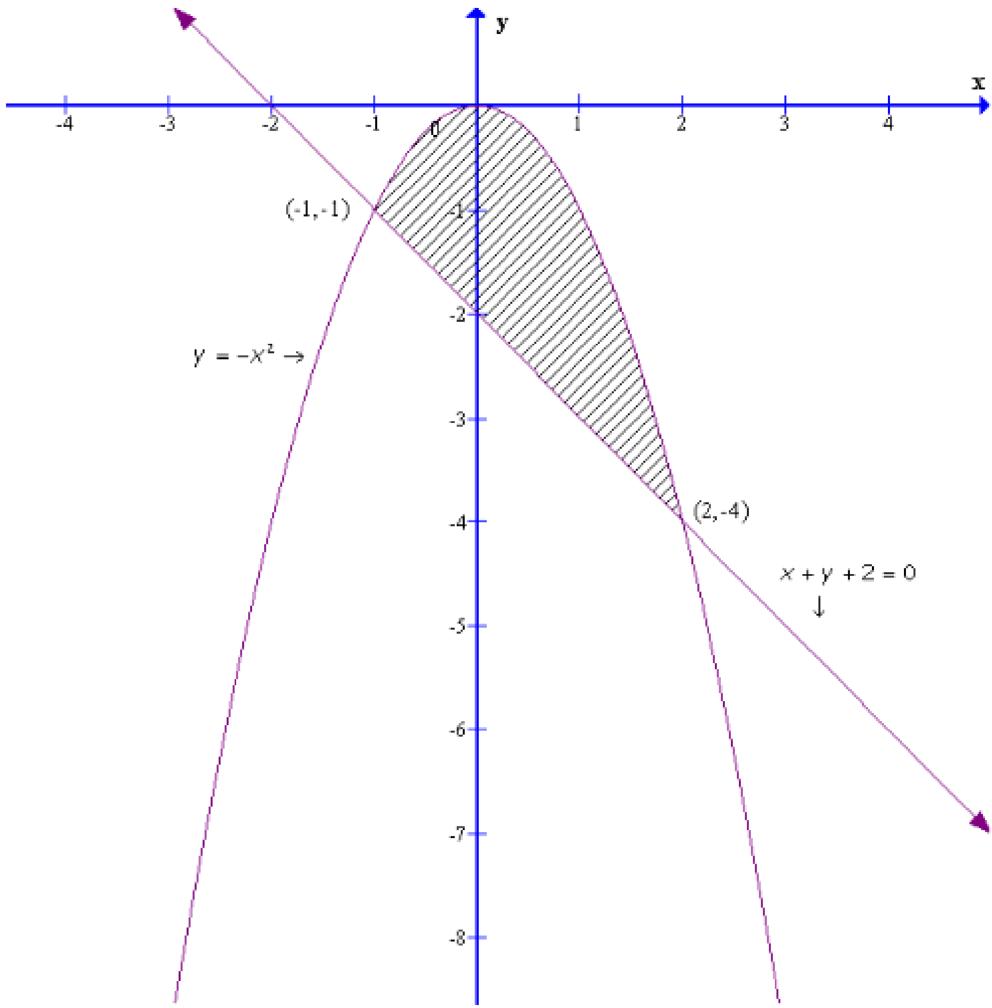
$$= \frac{1}{4} \left[3y^2 + 7y - \frac{y^3}{3} \right]_{-1}^{7}$$

$$= \frac{1}{4} \left[\left(147 + 49 - \frac{343}{3} \right) - \left(3 - 7 + \frac{1}{3} \right) \right]$$

$$= \frac{1}{4} \left[\frac{245}{3} + \frac{11}{3} \right]$$

$$A = \frac{64}{3} \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q28



Area of the bounded region

$$= \int_{-1}^{2} -x^2 - (-2-x)\, dx$$

$$= \left[-\frac{x^3}{3} + 2x + \frac{x^2}{2} \right]_{-1}^{2}$$

$$= \left[-\frac{8}{3} + 6 \right] - \left(\frac{1}{3} + \frac{1}{2} - 2 \right)$$

$$= \frac{9}{2} \text{ sq.units}$$

Areas of Bounded Regions Ex-21-3 Q29

To find area bounded by

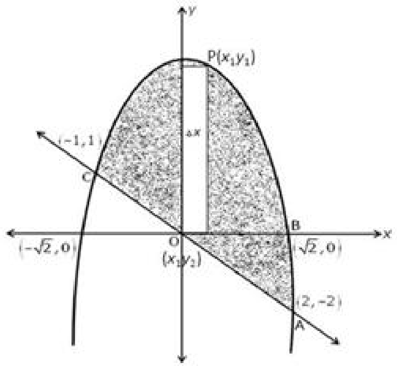
$y = 2 - x^2$ ---(1)

and $y + x = 0$ ---(2)

Equation (1) represents a parabola with vertex $(0,2)$ and downward, meets axes at $(\pm\sqrt{2},0)$.

Equation (2) represents a line passing through $(0,0)$ and $(2,-2)$. The points of intersection of line and parabola are $(2,-2)$ and $(-1,1)$.

A rough sketch of curves is as follows:-



Shaded region is sliced into rectangles with area $= (y_1 - y_2)\Delta x$. It slides from $x = -1$ to $x = 2$, so

Required area = Region $ABPCOA$

$$A = \int_{-1}^{2}(y_1 - y_2)dx$$

$$= \int_{-1}^{2}\left(2 - x^2 + x\right)dx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2}$$

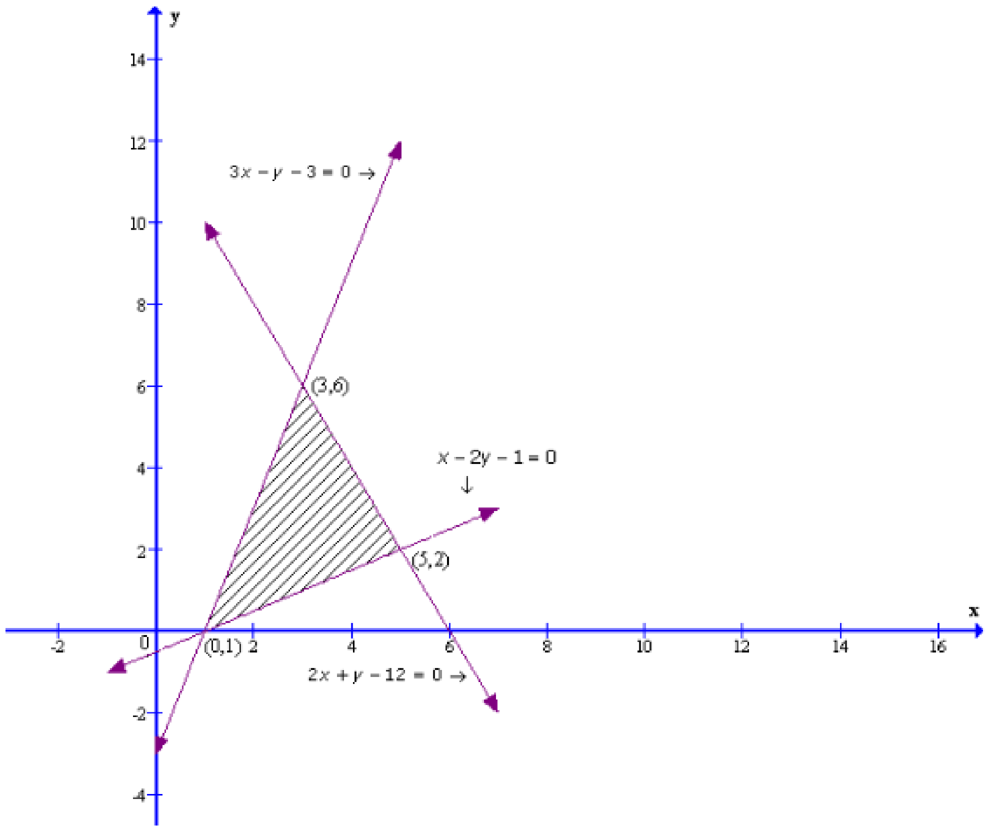
$$= \left[\left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)\right]$$

$$= \left[\frac{10}{3} + \frac{7}{6}\right]$$

$$= \frac{27}{6}$$

$$A = \frac{9}{2} \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q30



Area of the bounded region

$$= \int_0^3 3x - 3 - \left(\frac{x-1}{2}\right)dx + \int_3^5 12 - 2x - \left(\frac{x-1}{2}\right)dx$$

$$= \left[\frac{3x^2}{2} - 3x - \frac{x^2}{4} + \frac{1}{2}x\right]_0^3 + \left[12x - 2\frac{x^2}{2} - \frac{x^2}{4} + \frac{1}{2}x\right]_3^5$$

$$= \left[\frac{27}{2} - 9 - \frac{9}{4} + \frac{3}{2}\right] + \left[60 - 25 - \frac{25}{4} + \frac{5}{2} - 36 + 9 + \frac{9}{4} - \frac{3}{2}\right]$$

= 11 sq.units

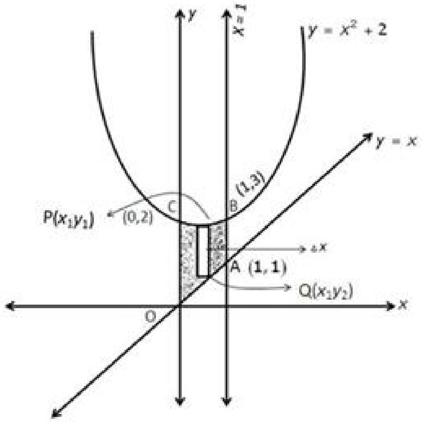
Areas of Bounded Regions Ex-21-3 Q31

To find area bounded by $x = 0$, $x = 1$
and

$y = x \qquad ---(1)$

$y = x^2 + 2 \qquad ---(2)$

Equation (1) is a line passing through $(2,2)$ and $(0,0)$. Equation (2) is a parabola upward with vertex at $(0,2)$. A rough sketch of curves is as under:-



Shaded region is sliced into rectangles of area $= (y_1 - y_2)\Delta x$. It slides from $x = 0$ to $x = 1$, so

Required area = Region $OABCO$

$$A = \int_0^1 (y_1 - y_2)dx$$

$$= \int_0^1 (x^2 + 2 - x)dx$$

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^1$$

$$= \left[\left(\frac{1}{3} + 2 - \frac{1}{2}\right) - (0)\right]$$

$$= \left(\frac{2 + 12 - 3}{6}\right)$$

$$A = \frac{11}{6} \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q32

To find area bounded by

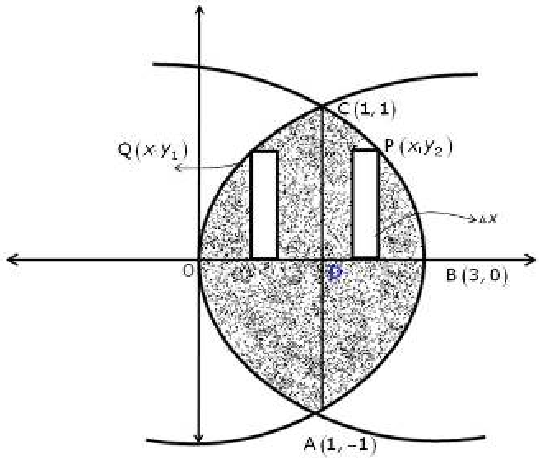
$x = y^2 \qquad ---(1)$

and

$x = 3 - 2y^2$

$2y^2 = -(x - 3) \qquad ---(2)$

Equation (1) represents an upward parabola with vertex $(0,0)$ and axis $-y$. Equation (2) represents a parabola with vertex $(3,0)$ and axis as x-axis. They intersect at $(1,-1)$ and $(1,1)$. A rough sketch of the curves is as under:-



Required area = Region $OABCO$

$A = 2$ Region $OBCO$

$= 2[\text{Region } ODCO + \text{Region } BDCB]$

$= 2\left[\int_0^1 y_1 dx + \int_1^3 y_2 dx\right]$

$= 2\left[\int_0^1 \sqrt{x}\,dx + \int_1^3 \sqrt{\frac{3-x}{2}}\,dx\right]$

$= 2\left[\left(\frac{2}{3}x\sqrt{x}\right)_0^1 + \left(\frac{2}{3}\cdot\left(\frac{3-x}{2}\right)\sqrt{\frac{3-x}{2}}\cdot(-2)\right)_1^3\right]$

$= 2\left[\left(\frac{2}{3} - 0\right) + \left\{(0) - \left(\frac{2}{3}.1.1.(-2)\right)\right]\right.$

$= 2\left[\frac{2}{3} + \frac{4}{3}\right]$

$A = 4$ sq. units

Areas of Bounded Regions Ex-21-3 Q33

To find area of $\triangle ABC$ with $A(4,1)$, $B(6,6)$ and $C(8,4)$.

Equation of AB,

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 1 = \left(\frac{6 - 1}{6 - 4}\right)(x - 4)$$

$$y - 1 = \frac{5}{2}x - 10$$

$$y = \frac{5}{2}x - 9 \qquad ---(1)$$

Equation of BC,

$$y - 6 = \left(\frac{4 - 6}{8 - 6}\right)(x - 6)$$

$$= -1(x - 6)$$

$$y = -x + 12 \qquad ---(2)$$

Equation of AC,

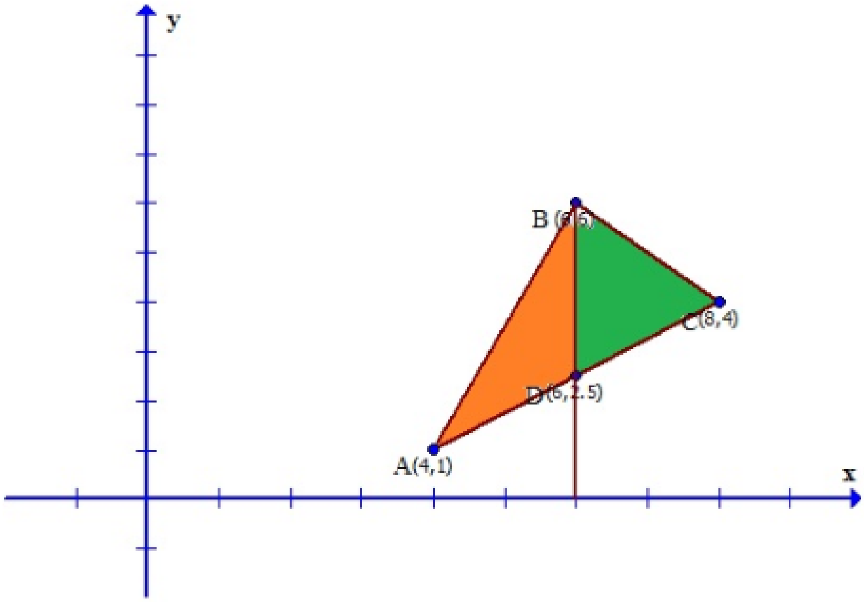
$$y - 1 = \left(\frac{4 - 1}{8 - 4}\right)(x - 4)$$

$$y - 1 = \frac{3}{4}(x - 4)$$

$$\Rightarrow \quad y = \frac{3}{4}x - 3 + 1$$

$$y = \frac{3}{4}x - 2 \qquad ---(3)$$

A rough sketch is as under:-



Clearly, Area of ΔABC = Area ADB + Area BDC

Area ADB: To find the area ADB, we slice it into vertical strips.

We observe that each vertical strip has its lower end on side AC and the upper end on AB. So the approximating rectangle has

Length $= y_2 - y_1$

Width $= \Delta x$

Area $= (y_2 - y_1)\Delta x$

Since the approximating rectangle can move from $x = 4$ to 6,

the area of the triangle ADB $= \int_4^6 (y_2 - y_1)dx$

$$\Rightarrow \text{area of the triangle ADB} = \int_4^6 \left[\left(\frac{5x}{2} - 9\right) - \left(\frac{3}{4}x - 2\right)\right]dx$$

$$\Rightarrow \text{area of the triangle ADB} = \int_4^6 \left(\frac{5x}{2} - 9 - \frac{3}{4}x + 2\right)dx$$

$$\Rightarrow \text{area of the triangle ADB} = \int_4^6 \left(\frac{7x}{4} - 7\right)dx$$

$$\Rightarrow \text{area of the triangle ADB} = \left(\frac{7x^2}{4 \times 2} - 7x\right)_4^6$$

$$\Rightarrow \text{area of the triangle ADB} = \left(\frac{7 \times 36}{8} - 7 \times 6\right) - \left(\frac{7 \times 16}{8} - 7 \times 4\right)$$

$$\Rightarrow \text{area of the triangle } ADB = \left(\frac{63}{2} - 42 - 14 + 28\right)$$

$$\Rightarrow \text{area of the triangle } ADB = \left(\frac{63}{2} - 28\right)$$

$$\text{Similarly, Area } BDC = \int_6^8 (y_4 - y_3)dx$$

$$\Rightarrow \text{Area } BDC = \int_6^8 (y_4 - y_3)dx$$

$$\Rightarrow \text{Area } BDC = \int_6^8 \left[(-x + 12) - \left(\frac{3}{4}x - 2\right)\right]dx$$

$$\Rightarrow \text{Area } BDC = \int_6^8 \left[\frac{-7x}{4} + 14\right]dx$$

$$\Rightarrow \text{Area } BDC = \left[-\frac{7x^2}{8} + 14x\right]_6^8$$

$$\Rightarrow \text{Area } BDC = \left[-\frac{7 \times 64}{8} + 14 \times 8\right] - \left[-\frac{7 \times 36}{8} + 14 \times 6\right]$$

$$\Rightarrow \text{Area } BDC = \left[-56 + 112 + \frac{63}{2} - 84\right]$$

$$\Rightarrow \text{Area } BDC = \left(\frac{63}{2} - 28\right)$$

$$\text{Thus, Area } ABC = \text{Area } ADB + \text{Area } BDC$$

$$\Rightarrow \text{Area } ABC = \left(\frac{63}{2} - 28\right) + \left(\frac{63}{2} - 28\right)$$

$$\Rightarrow \text{Area } ABC = 63 - 56$$

$$\Rightarrow \text{Area } ABC = 7 \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q34

To find area of region

$$\left\{(x, y) : |x - 1| \le y \le \sqrt{5 - x^2}\right\}$$

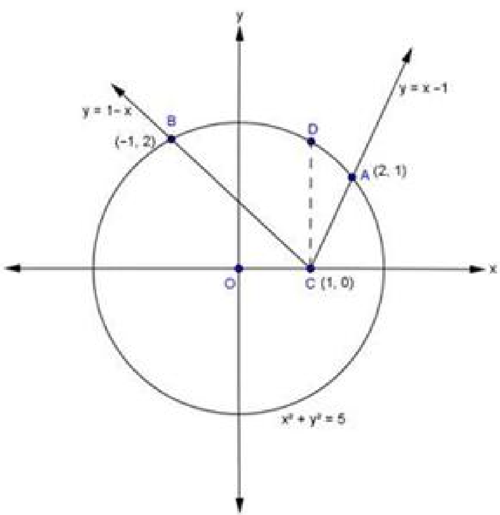
$\Rightarrow \quad |x - 1| = y$

$$\Rightarrow \quad y = \begin{cases} 1 - x, \text{ if } x < 1 & ---(1) \\ x - 1, \text{ if } x \ge 1 & ---(2) \end{cases}$$

And $\quad x^2 + y^2 = 5 \qquad ---(3)$

Equation (1) and (2) represent straight lines and equation (3) is a circle with centre $(0,0)$, meets axes at $(\pm\sqrt{5}, 0)$ and $(0, \pm\sqrt{5})$.

A rough sketch of the curves is as under:



Shaded region represents the required area.

Required area = Region $BCDB$ + Region $CADC$

$$A = \int_{-1}^{1}(y_1 - y_2)\,dx + \int_{1}^{2}(y_1 - y_2)\,dx$$

$$= \int_{-1}^{1}\left[\sqrt{5-x^2} - 1 + x\right]dx + \int_{1}^{2}\left(\sqrt{5-x^2} - x + 1\right)dx$$

$$= \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}} - x + \frac{x^2}{2}\right]_{-1}^{1} + \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}} - \frac{x^2}{2} + x\right]_{1}^{2}$$

$$= \left[\left(\frac{1}{2}.2 + \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) - 1 + \frac{1}{2}\right) - \left(-\frac{1}{2}.2 - \frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + 1 + \frac{1}{2}\right)\right]$$

$$+ \left[\left(1.1. + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - 2 + 2\right) - \left(\frac{1}{2}.2 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2} + 1\right)\right]$$

$$= \left[1 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2} + 1 + \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{3}{2}\right] + \left[1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - 1 - \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2}\right]$$

$$= 5\sin^{-1}\frac{1}{\sqrt{5}} + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}} - \frac{5}{2}\sin^{-1}\frac{1}{\sqrt{5}} - \frac{1}{2}$$

$$A = \left[\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right) - \frac{1}{2}\right] \text{ sq. units.}$$

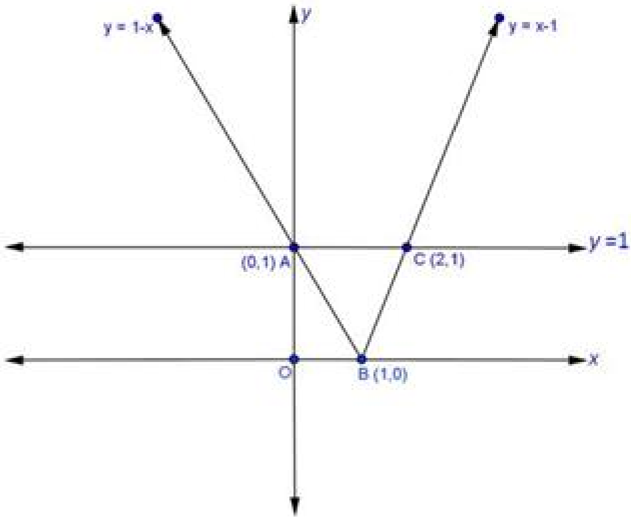
Areas of Bounded Regions Ex-21-3 Q35

To find area bounded by $y = 1$ and

$$y = |x - 1|$$

$$y = \begin{cases} x - 1, \text{ if } x \ge 0 & \text{---}(1) \\ 1 - x, \text{ if } x < 0 & \text{---}(2) \end{cases}$$

A rough sketch of the curve is as under:-



Shaded region is the required area. So

Required area = Region $ABCA$

$$A = \text{Region } ABDA + \text{Region } BCDB$$

$$= \int_0^1 (y_1 - y_2)\,dx + \int_1^2 (y_1 - y_3)\,dx$$

$$= \int_0^1 (1 - 1 + x)\,dx + \int_1^2 (1 - x + 1)\,dx$$

$$= \int_0^1 x\,dx + \int_1^2 (2 - x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left[(4 - 2) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + \left(2 - 2 + \frac{1}{2}\right)$$

$A = 1$ sq. unit

Areas of Bounded Regions Ex-21-3 Q36

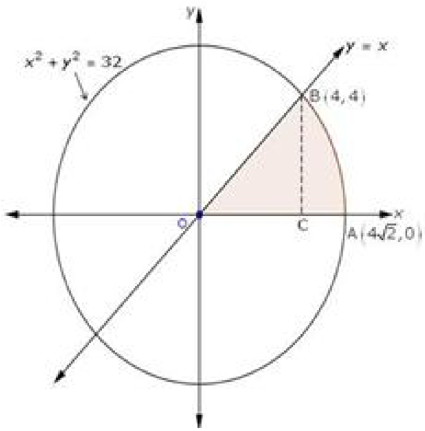
To find area of in first quadrant enclosed by x-axis, the line $y = x$ and circle

$$x^2 + y^2 = 32 \qquad ---(1)$$

Equation (1) is a circle with centre $(0,0)$ and meets axes at $(\pm 4\sqrt{2}, 0), (0, \pm 4\sqrt{2})$.

And $y = x$ is a line passes through $(0,0)$ and intersect circle at $(4,4)$.

A rough sketch of curve is as under:-



Required area is shaded region $OABO$

Region $OABO$ = Region $OCBO$ + Region $CABC$

$$= \int_0^4 y_1 dx + \int_4^{4\sqrt{2}} y_2 dx$$
$$= \int_0^4 x dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2} dx$$
$$= \left(\frac{x^2}{2}\right)_0^4 + \left[\frac{x}{2}\sqrt{32-x^2} + \frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$
$$= (8 - 0) + \left[\left(0 + 16.\frac{\pi}{2}\right) - \left(8 + 16.\frac{\pi}{4}\right)\right]$$

$$= 8 + 8\pi - 8 - 4\pi$$

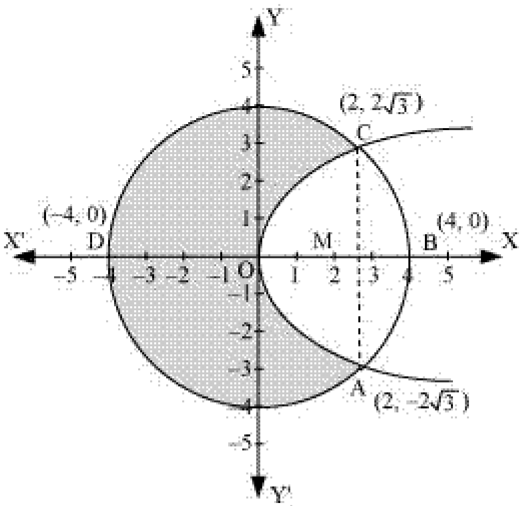
$A = 4\pi$ sq. units

Areas of Bounded Regions Ex-21-3 Q37

The given equations are

$x^2+y^2=16$ … (1)

$y^2=6x$ … (2)



Area bounded by the circle and parabola

$$=2\left[\text{Area}(\text{OADO})+\text{Area}(\text{ADBA})\right]$$

$$=2\left[\int_0^2\sqrt{16x}\,dx+\int_2^4\sqrt{16-x^2}\,dx\right]$$

$$=2\left[\sqrt{6}\left\{\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\}_0^2\right]+2\left[\frac{x}{2}\sqrt{16-x^2}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$=2\sqrt{6}\times\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^2+2\left[8\cdot\frac{\pi}{2}-\sqrt{16-4}-8\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$=\frac{4\sqrt{6}}{3}\left(2\sqrt{2}\right)+2\left[4\pi-\sqrt{12}-8\frac{\pi}{6}\right]$$

$$= \frac{16\sqrt{3}}{3} + 8\pi - 4\sqrt{3} - \frac{8}{3}\pi$$

$$= \frac{4}{3}\left[4\sqrt{3} + 6\pi - 3\sqrt{3} - 2\pi\right]$$

$$= \frac{4}{3}\left[\sqrt{3} + 4\pi\right]$$

$$= \frac{4}{3}\left[4\pi + \sqrt{3}\right] \text{ square units}$$

$$\text{Area of circle} = \pi\,(r)^2$$

$$= \pi\,(4)^2 = 16\pi \text{ square units}$$

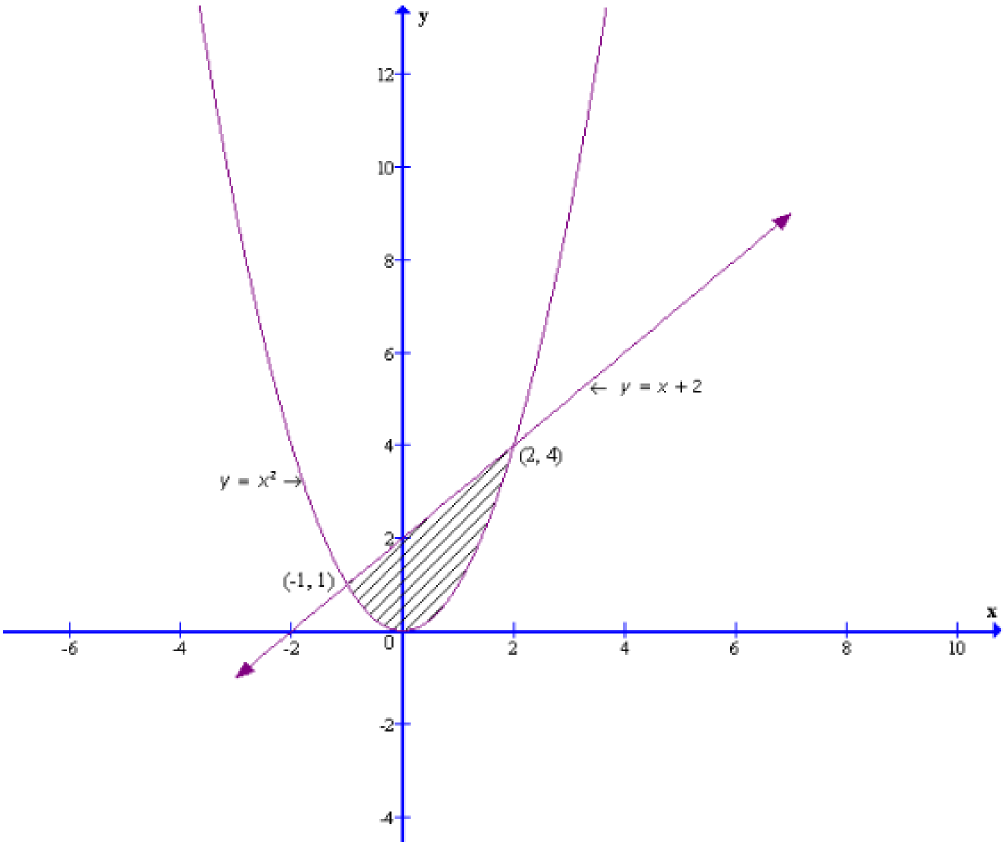
Thus, Required area $= 16\pi - \frac{4}{3}\left[4\pi + \sqrt{3}\right]$

$$= \frac{4}{3}\left[4 \times 3\pi - 4\pi - \sqrt{3}\right]$$

$$= \frac{4}{3}\left(8\pi - \sqrt{3}\right)$$

$$= \left(\frac{32}{3}\pi - \frac{4\sqrt{3}}{3}\right) \text{sq. units}$$

Areas of Bounded Regions Ex-21-3 Q38



Area of the bounded region

$$= \int_{-1}^{2} x+2-x^2 \, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{9}{2} \text{ sq.units}$$

Areas of Bounded Regions Ex-21-3 Q39

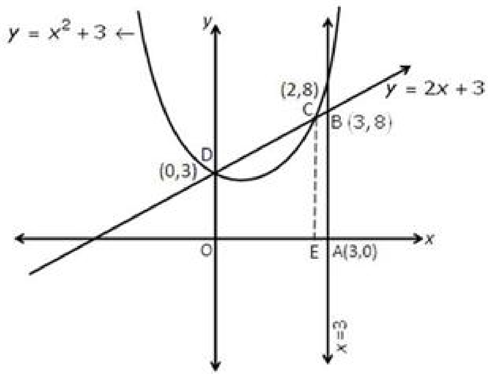
To find area of region

$\{(x,y) : 0 \le y \le x^2 + 3,\ 0 \le y \le 2x + 3,\ 0 \le x \le 3\}$

$$\Rightarrow \quad y = x^2 + 3 \qquad ---(1)$$
$$y = 2x + 3 \qquad ---(2)$$

and $\quad x = 0, x = 3$

Equation (1) represents a parabola with vertex $(3,0)$ and axis as y-axis. Equation (2) represents a line a passing through $(0,3)$ and $\left(-\frac{3}{2},0\right)$, a rough sketch of curve is as under:-



Required area = Region $ABCDOA$

A = Region $ABCEA$ + Region $ECDOE$

$$= \int_2^3 y_1 dx + \int_0^2 y_2 dx$$
$$= \int_2^3 (2x + 3)\,dx + \int_0^2 (x^2 + 3)\,dx$$
$$= \left(x^2 + 3x\right)_2^3 + \left(\frac{x^3}{3} + x\right)_0^2$$
$$= [(9 + 9) - (4 + 6)] + \left[\left(\frac{8}{3} + 2\right) - (0)\right]$$
$$= [18 - 10] + \left[\frac{14}{3}\right]$$
$$= 8 + \frac{14}{3}$$

$A = \frac{38}{3}$ sq. units

Areas of Bounded Regions Ex-21-3 Q40

To find area bounded by positive x-axis and curve

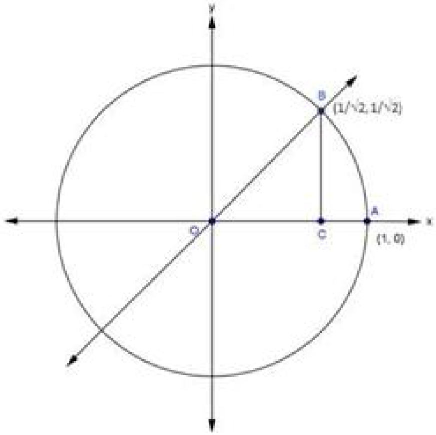
$y = \sqrt{1 - x^2}$

$x^2 + y^2 = 1 \qquad ---(1)$

$x = y \qquad ---(2)$

Equation (1) represents a circle with centre $(0,0)$ and meets axes at $(\pm 1, 0), (0, \pm 1)$.

Equation (2) represents a line passing through $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and they are also points of intersection. A rough sketch of the curve is as under:-



Required area = Region $OABO$

A = Region $OCBO$ + Region $CABC$

$$= \int_0^{\frac{1}{\sqrt{2}}} y_1 dx + \int_{\frac{1}{\sqrt{2}}}^{1} y_2 dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x dx + \int_{\frac{1}{\sqrt{2}}}^{1} \sqrt{1 - x^2} dx$$

$$= \left[\frac{x^2}{2}\right]_0^{\frac{1}{\sqrt{2}}} + \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= \left[\frac{1}{4} - 0\right] + \left[\left(0 + \frac{1}{2}.\frac{\pi}{2}\right) - \left(\frac{1}{2\sqrt{2}}.\frac{1}{\sqrt{2}} + \frac{1}{2}.\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{\pi}{4} - \frac{1}{4} - \frac{\pi}{8}$$

$$A = \frac{\pi}{8} \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q41

To find area bounded by lines

$y = 4x + 5$ (Say AB) $\quad ---(1)$

$y = 5 - x$ (Say BC) $\quad ---(2)$

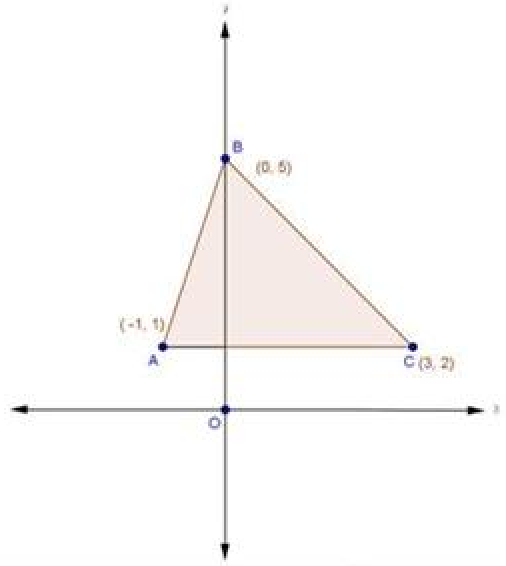
$4y = x + 5$ (Say AC) $\quad ---(3)$

By solving equation (1) and (2), we get $B(0,5)$

By solving equation (2) and (3), we get $C(3,2)$

By solving equation (1) and (3), we get $A(-1,1)$

A rough sketch of the curve is as under:-



Shaded area $\triangle ABC$ is the required area.

Required area $= ar(\triangle ABD) + ar(\triangle BDC) \quad ---(1)$

$$ar(\triangle ABD) = \int_{-1}^{0}(y_1 - y_3)\,dx$$

$$= \int_{-1}^{0}\left(4x + 5 - \frac{x}{4} - \frac{5}{4}\right)dx$$

$$= \int_{-1}^{0}\left(\frac{15x}{4} + \frac{15}{4}\right)dx$$

$$= \frac{15}{4}\left(\frac{x^2}{2} + x\right)_{-1}^{0}$$

$$= \frac{15}{4}\left[(0) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{15}{4} \times \frac{1}{2}$$

$$ar(\triangle ABD) = \frac{15}{8} \text{ sq. units} \qquad ---(2)$$

$$ar(\triangle BDC) = \int_0^3 (y_2 - y_3)\,dx$$

$$= \int_0^3 \left[(5 - x) - \left(\frac{x}{4} + \frac{5}{4}\right)\right] dx$$

$$= \int_0^3 \left[5 - x - \frac{x}{4} - \frac{5}{4}\right] dx$$

$$= \int_0^3 \left(\frac{-5x}{4} + \frac{15}{4}\right) dx$$

$$= \frac{5}{4}\left(3x - \frac{x^2}{2}\right)$$

$$= \frac{5}{4}\left(9 - \frac{9}{2}\right)$$

$$ar(\triangle BDC) = \frac{45}{8} \text{ sq. units} \qquad ---(3)$$

Using equation (1), (2) and (3),

$$ar(\triangle ABC) = \frac{15}{8} + \frac{45}{8}$$

$$= \frac{60}{8}$$

$$ar(\triangle ABC) = \frac{15}{2} \text{ sq. units}$$

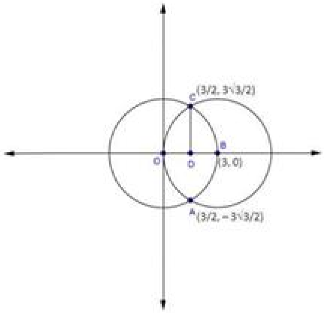
Areas of Bounded Regions Ex-21-3 Q42

To find area enclosed by

$x^2 + y^2 = 9 \qquad ---(1)$

$(x-3)^2 + y^2 = 9 \qquad ---(2)$

Equation (1) represents a circle with centre $(0,0)$ and meets axes at $(\pm 3,0),(0,\pm 3)$.
Equation (2) is a circle with centre $(3,0)$ and meets axes at $(0,0),(6,0)$.
they intersect each other at $\left(\frac{3}{2},\frac{3\sqrt{3}}{2}\right)$ and $\left(\frac{3}{2},-\frac{3\sqrt{3}}{2}\right)$. A rough sketch of the curves is as under:



Shaded region is the required area.

Required area = Region $OABCO$

$$A = 2(\text{Region } OBCO)$$
$$= 2(\text{Region } ODCO + \text{Region } DBCD)$$
$$= 2\left[\int_0^{\frac{3}{2}} \sqrt{9-(x-3)^2}\,dx + \int_{\frac{3}{2}}^{3} \sqrt{9-x^2}\,dx\right]$$
$$= 2\left[\left\{\frac{(x-3)}{2}\sqrt{9-(x-3)^2} + \frac{9}{2}\sin^{-1}\frac{(x-3)}{3}\right\}_0^{\frac{3}{2}} + \left\{\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right\}_{\frac{3}{2}}^{3}\right]$$
$$= 2\left[\left\{\left(-\frac{3}{4}\sqrt{9-\frac{9}{4}} + \frac{9}{2}\sin^{-1}\left(-\frac{3}{6}\right)\right) - \left(0 + \frac{9}{2}\sin^{-1}(-1)\right)\right\} + \left\{\left(0 + \frac{9}{2}\sin^{-1}(1)\right) - \left(\frac{3}{4}\sqrt{9-\frac{9}{4}} + \frac{9}{2}\sin^{-1}\left(\frac{1}{2}\right)\right)\right\}\right]$$
$$= 2\left[\left\{-\frac{9\sqrt{3}}{8} - \frac{9}{2}\cdot\frac{\pi}{6} + \frac{9}{2}\cdot\frac{\pi}{2}\right\} + \left\{\frac{9}{2}\cdot\frac{\pi}{2} - \frac{9\sqrt{3}}{8} - \frac{9}{2}\cdot\frac{\pi}{6}\right\}\right]$$
$$= 2\left[-\frac{9\sqrt{3}}{8} - \frac{3\pi}{4} + \frac{9\pi}{4} + \frac{9\pi}{4} - \frac{9\sqrt{3}}{8} - \frac{3\pi}{4}\right]$$
$$= 2\left[\frac{12\pi}{4} - \frac{18\sqrt{3}}{8}\right]$$
$$A = \left(6\pi - \frac{9\sqrt{3}}{2}\right) \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q43

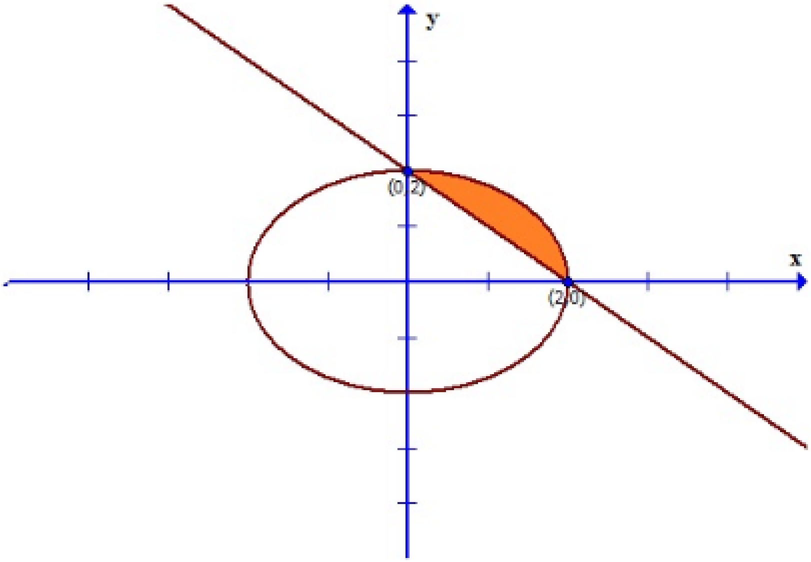
The equation of the given curves are

$x^2 + y^2 = 4 \ldots.(1)$

$x + y = 2 \ldots\ldots.(2)$

Clearly $x^2 + y^2 = 4$ represents a circle and $x + y = 2$ is the equation of a straight line cutting x and y axes at (0,2) and (2,0) respectively.

The smaller region bounded by these two curves is shaded in the following figure.



Length $= y_2 - y_1$

Width $= \Delta x$ and

Area $= (y_2 - y_1)\Delta x$

Since the approximating rectangle can move from $x = 0$ to $x = 2$, the required area is given by

$$A = \int_0^2 (y_2 - y_1)dx$$

We have $y_1 = 2 - x$ and $y_2 = \sqrt{4 - x^2}$

Thus,

$$A = \int_0^2 \left(\sqrt{4 - x^2} - 2 + x\right)dx$$

$$\Rightarrow A = \int_0^2 \left(\sqrt{4 - x^2}\right)dx - 2\int_0^2 dx + \int_0^2 xdx$$

$$\Rightarrow A = \left[\frac{x\sqrt{4 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 - 2(x)_0^2 + \left(\frac{x^2}{2}\right)_0^2$$

$$\Rightarrow A = \frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right) - 4 + 2$$

$$\Rightarrow A = 2\sin^{-1}(1) - 2$$

$$\Rightarrow A = 2 \times \frac{\pi}{2} - 2$$

$$\Rightarrow A = \pi - 2 \text{ sq.units}$$

Areas of Bounded Regions Ex-21-3 Q44

To find area of region

$$\left\{(x, y): \frac{x^2}{9}+\frac{y^2}{4} \leq 1 \leq \frac{x}{3}+\frac{y}{2}\right\}$$

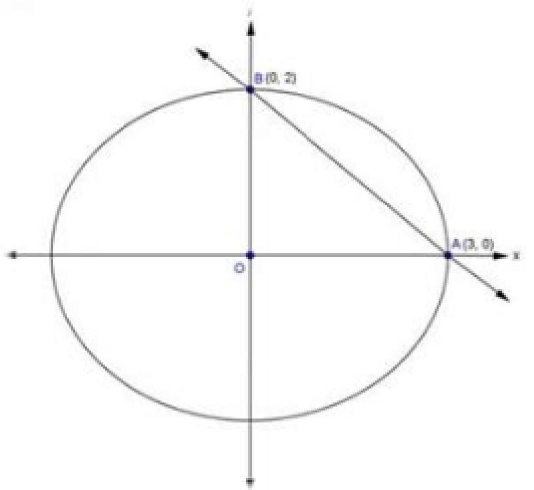
Here

$$\frac{x^2}{9}+\frac{y^2}{4}=1 \qquad ---(1)$$

$$\frac{x}{3}+\frac{y}{2}=1 \qquad ---(2)$$

Equation (1) represents an ellipse with centre at origin and meets axes at $(\pm 3,0)$, $(0,\pm 2)$. Equation (2) is a line that meets axes at $(3,0),(0,2)$.

A rough sketch is as under:



Shaded region represents required area. This is sliced into rectangles with area $(y_1-y_2)\Delta x$ which slides from $x=0$ to $x=3$, so

Required area = Region $APBQA$

$$A=\int_0^3 (y_1-y_2)\,dx$$

$$=\int_0^3\left[\frac{2}{3}\sqrt{9-x^2}\,dx-\frac{2}{3}(3-x)\,dx\right]$$

$$=\frac{2}{3}\left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)-3x+\frac{x^2}{2}\right]_0^3$$

$$=\frac{2}{3}\left[\left\{0+\frac{9}{2}.\frac{\pi}{2}-9+\frac{9}{2}\right\}-\{0\}\right]$$

$$=\frac{2}{3}\left[\frac{9\pi}{4}-\frac{9}{2}\right]$$

$$A=\left(\frac{3\pi}{2}-3\right) \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q45

To find area enclosed by

$$y = |x - 1|$$

$$\Rightarrow \quad y = \begin{cases} -(x-1), \text{ if } x - 1 < 0 \\ (x-1), \text{ if } x - 1 \geq 0 \end{cases}$$

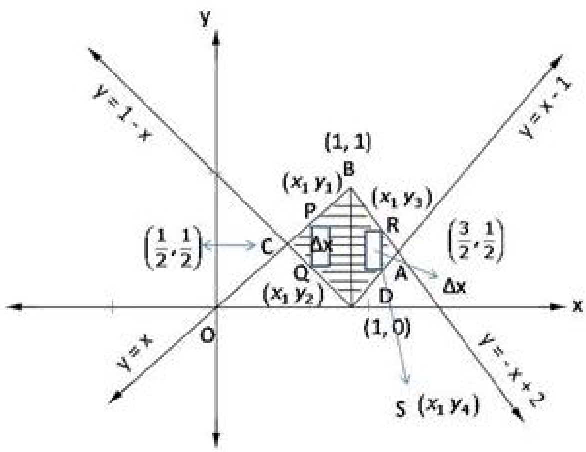
$$\Rightarrow \quad y = \begin{cases} 1 - x, \text{ if } x < 1 & ---(1) \\ x - 1, \text{ if } x \geq 1 & ---(2) \end{cases}$$

And $\quad y = -|x - 1| + 1$

$$\Rightarrow \quad y = \begin{cases} +(x-1)+1, \text{ if } x - 1 < 0 \\ -(x-1)+1, \text{ if } x - 1 \geq 0 \end{cases}$$

$$y = \begin{cases} x, \quad \text{ if } x < 1 & ---(3) \\ -x + 2, \text{ if } x \geq 1 & ---(4) \end{cases}$$

A rough sketch of equation of lines $(1),(2),(3),(4)$ is given as:



Shaded region is the required area.

Required area = Region $ABCDA$

Required area = Region $BDCB$ + Region $ABDA$ $\quad ---(1)$

Region $BDCB$ is sliced into rectangles of area = $(y_1 - y_2)\Delta x$ and it slides from

$x = \frac{1}{2}$ to $x = 1$

Region $ABDA$ is sliced into rectangle of area $= (y_3 - y_4)\Delta x$ and it slides from $x = 1$ to $x = \frac{3}{2}$. So, using equation (1),

Required area = Region $BDCB$ + Region $ABDA$

$$= \int_{\frac{1}{2}}^{1} (y_1 - y_2)\,dx + \int_{1}^{\frac{3}{2}} (y_3 - y_4)\,dx$$

$$= \int_{\frac{1}{2}}^{1} (x - 1 + x)\,dx + \int_{1}^{\frac{3}{2}} (-x + 2 - x + 1)\,dx$$

$$= \int_{\frac{1}{2}}^{1} (2x - 1)\,dx + \int_{1}^{\frac{3}{2}} (3 - 2x)\,dx$$

$$= \left[x^2 - x\right]_{\frac{1}{2}}^{1} + \left[3x - x^2\right]_{1}^{\frac{3}{2}}$$

$$= \left[(1 - 1) - \left(\frac{1}{4} - \frac{1}{2}\right)\right] + \left[\left(\frac{9}{2} - \frac{9}{4}\right) - (3 - 1)\right]$$

$$= \frac{1}{4} + \frac{9}{4} - 2$$

$A = \frac{1}{2}$ sq.units

Areas of Bounded Regions Ex-21-3 Q46

To find area enclosed by

$$3x^2 + 5y = 32$$

$$3x^2 = -5\left(y - \frac{32}{5}\right) \qquad ---(1)$$

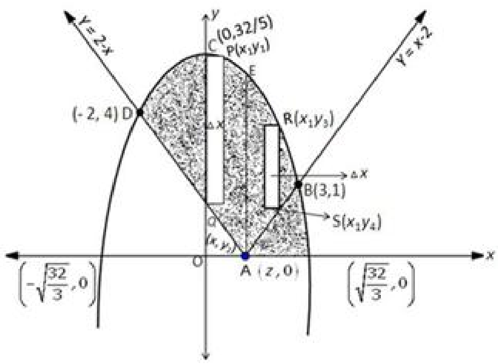
And

$$y = |x - 2|$$

$$\Rightarrow \qquad y = \begin{cases} -(x-2), \text{ if } x - 2 < 1 \\ (x-2), \text{ if } x - 2 \geq 1 \end{cases}$$

$$\Rightarrow \qquad y = \begin{cases} 2 - x, \text{ if } x < 2 \\ x - 2, \text{ if } x \geq 2 \end{cases} \qquad ---(2)$$

Equation (1) represents a downward parabola with vertex $\left(0, \frac{32}{5}\right)$ and equation (2) represents lines. A rough sketch of curves is given as:-



Required area = Region $ABECDA$

$$A = \text{Region } ABEA + \text{Region } AECDA$$

$$= \int_2^3 (y_3 - y_4)\,dx + \int_{-2}^2 (y_1 - y_2)\,dx$$

$$= \int_2^3 \left(\frac{32 - 3x^2}{5} - x + 2\right)dx + \int_{-2}^2 \left(\frac{32 - 3x^2}{5} - 2 + x\right)dx$$

$$= \int_2^3 \left(\frac{32 - 3x^2 - 5x + 10}{5}\right)dx + \int_{-2}^2 \left(\frac{32 - 3x^2 - 10 + 5x}{5}\right)dx$$

$$= \frac{1}{5}\left[\int_2^3 \left(42 - 3x^2 - 5x\right)dx + \int_{-2}^2 \left(22 - 3x^2 + 5x\right)dx\right]$$

$$A = \frac{1}{5}\left[\left(42x - x^3 - \frac{5x^2}{2}\right)_2^3 + \left(22x - x^3 + \frac{5x^2}{2}\right)_{-2}^2\right]$$

$$= \frac{1}{5}\left[\left\{\left(126 - 27 - \frac{45}{2}\right) - (84 - 8 - 10)\right\} + \left\{(44 - 8 + 10) - (-44 + 8 + 10)\right\}\right]$$

$$= \frac{1}{5}\left[\left\{\frac{153}{2} - 66\right\} + \{46 + 26\}\right]$$

$$= \frac{1}{5}\left[\frac{21}{2} + 72\right]$$

$$A = \frac{33}{2} \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q47

To area enclosed by

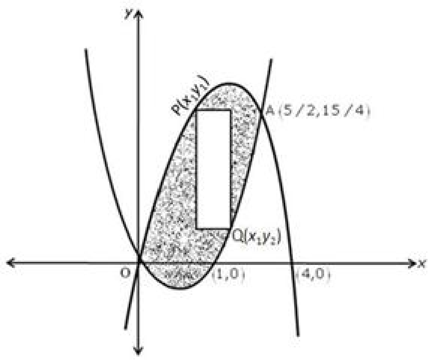
$$y = 4x - x^2$$
$$\Rightarrow \quad -y = x^2 - 4x + 4 - 4$$
$$\Rightarrow \quad -y + 4 = (x-2)^2$$
$$\Rightarrow \quad -(y-4) = (x-2)^2 \qquad ---(1)$$

and $y = x^2 - x$

$$\left(y + \frac{1}{4}\right) = \left(x - \frac{1}{2}\right)^2 \qquad ---(2)$$

Equation (1) represents a parabola downward with vertex at $(2,4)$ and meets axes at $(4,0),(0,0)$. Equation (2) represents a parabola upword whose vertex is $\left(\frac{1}{2}, -\frac{1}{4}\right)$ and meets axes at $(1,0),(0,0)$. Points of intersection of parabolas are $(0,0)$ and $\left(\frac{5}{2}, \frac{15}{4}\right)$.

A rough sketch of the curves is as under:-



Shaded region is required area it is sliced into rectangles with area $= (y_1 - y_2)\Delta x$. It slides from $x = 0$ to $x = \frac{5}{2}$, so

Required area = Region $OQAP$

$$A = \int_0^{\frac{5}{2}} (y_1 - y_2)dx$$
$$= \int_0^{\frac{5}{2}} \left[4x - x^2 - x^2 + x\right]dx$$
$$= \int_0^{\frac{5}{2}} \left[5x - 2x^2\right]dx$$
$$= \left[\frac{5x^2}{2} - \frac{2}{3}x^3\right]_0^{\frac{5}{2}}$$
$$= \left[\left(\frac{125}{8} - \frac{250}{24}\right) - (0)\right]$$
$$A = \frac{125}{24} \text{ sq. units}$$

Areas of Bounded Regions Ex-21-3 Q48

Given curves are

$y = 4x - x^2$

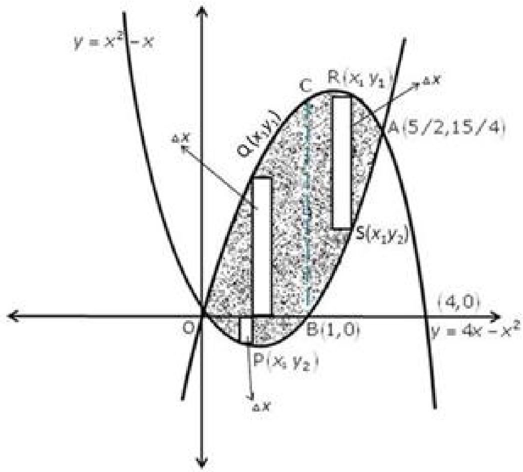
$\Rightarrow \quad -(y-4) = (x-2)^2 \qquad ---(1)$

and

$y = x^2 - x$

$\Rightarrow \quad \left(y + \frac{1}{4}\right)^2 = \left(x - \frac{1}{2}\right)^2 \qquad ---(2)$

Equation (1) represents a parabola downward with vertex at $(2,4)$ and meets axes at $(4,0),(0,0)$. Equation (2) represents a parabola upward whose vertex is $\left(\frac{1}{2}, -\frac{1}{4}\right)$ and meets axes at $(1,0),(0,0)$ and $\left(\frac{5}{2}, \frac{15}{4}\right)$. A rough sketch of the curves is as under:-



Area of the region above x-axis

A_1 = Area of region $OBACO$

= Region $OBCO$ + Region $BACB$

$$= \int_0^1 y_1 dx + \int_1^{\frac{5}{2}} (y_1 - y_2) dx$$

$$= \int_0^1 (4x - x^2) dx + \int_1^{\frac{5}{2}} (4x - x^2 - x^2 + x) dx$$

$$= \left(\frac{4x^2}{2} - \frac{x^3}{3}\right)_0^1 + \left[\frac{5x^2}{2} - \frac{2x^3}{3}\right]_1^{\frac{5}{2}}$$

$$= \left(2 - \frac{1}{3}\right) + \left[\left(\frac{125}{8} - \frac{250}{24}\right) - \left(\frac{5}{2} - \frac{2}{3}\right)\right]$$

$$= \frac{5}{3} + \frac{125}{24} - \frac{11}{6}$$

$$= \frac{121}{24} \text{ sq. units}$$

Area of the region below x-axis

A_2 = Area of region $OPBO$

= Region $OBCO$ + Region $BACB$

$$= \left|\int_0^1 y_2 dx\right|$$

$$= \left|\int_0^1 (x^2 - x) dx\right|$$

$$= \left|\left(\frac{x^3}{3} - \frac{x^2}{2}\right)_0^1\right|$$

$$= \left|\left(\frac{1}{3} - \frac{1}{2}\right) - (0)\right|$$

$$= \left|-\frac{1}{6}\right|$$

$$A_2 = \frac{1}{6} \text{ sq. units}$$

$$A_1 : A_2 = \frac{121}{24} : \frac{1}{6}$$

$$\Rightarrow \quad A_1 : A_2 = \frac{121}{24} : \frac{4}{24}$$

$$\Rightarrow \quad A_1 : A_2 = 121 : 4$$

Areas of Bounded Regions Ex-21-3 Q49

To find area bounded by the curve

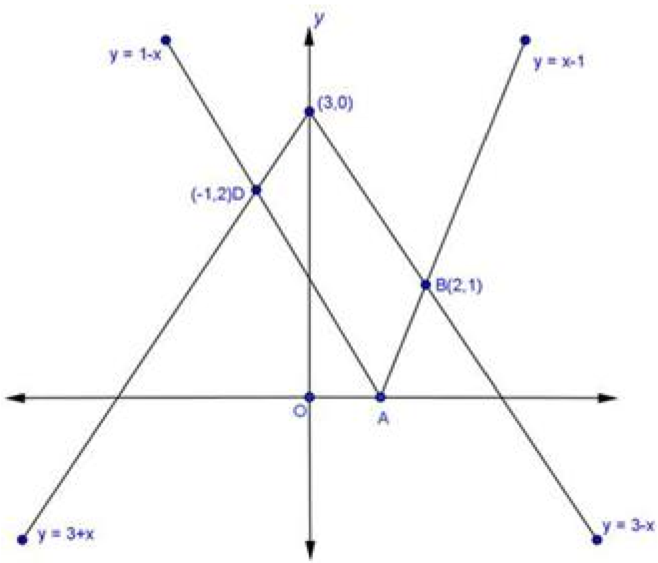
$$y = |x-1|$$

$$\Rightarrow \quad y = \begin{cases} 1-x, \text{ if } x < 1 & ---(1) \\ x-1, \text{ if } x \geq 1 & ---(2) \end{cases}$$

and $y = 3 - |x|$

$$\Rightarrow \quad y = \begin{cases} 3+x, \text{ if } x < 0 & ---(3) \\ 3-x, \text{ if } x \geq 0 & ---(4) \end{cases}$$

Drawing the rough sketch of lines (1), (2), (3) and (4) as under:-



Shaded region is the required area

Required area = Region $ABCDA$

$$A = \text{Region } ABFA + \text{Region } AFCEA + \text{Region } CDEC$$
$$= \int_1^2 (y_1 - y_2)\,dx + \int_0^1 (y_1 - y_3)\,dx + \int_{-1}^0 (y_4 - y_3)\,dx$$
$$= \int_1^2 (3 - x - x + 1)\,dx + \int_0^1 (3 - x - 1 + x)\,dx + \int_{-1}^0 (3 + x - 1 + x)\,dx$$

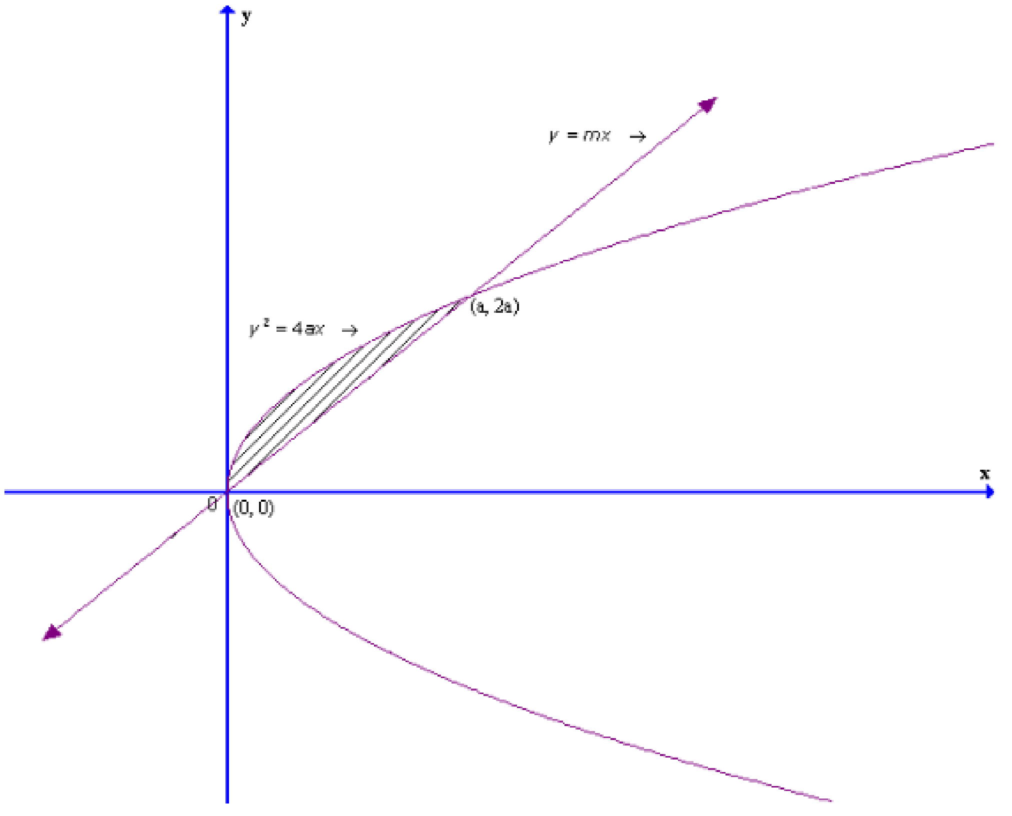
$$= \int_1^2 (4 - 2x)\,dx + \int_0^1 2\,dx + \int_{-1}^0 (2 + 2x)\,dx$$

$$= \left[4x - x^2\right]_1^2 + \left[2x\right]_0^1 + \left[2x + x^2\right]_{-1}^0$$
$$= \left[(8-4) - (4-1)\right] + [2-0] + \left[(0) - (-2+1)\right]$$

$$= (4-3) + 2 + 1$$

$A = 4$ sq. unit

Areas of Bounded Regions Ex-21-3 Q50



Area of the bounded region = $\frac{a^2}{12}$

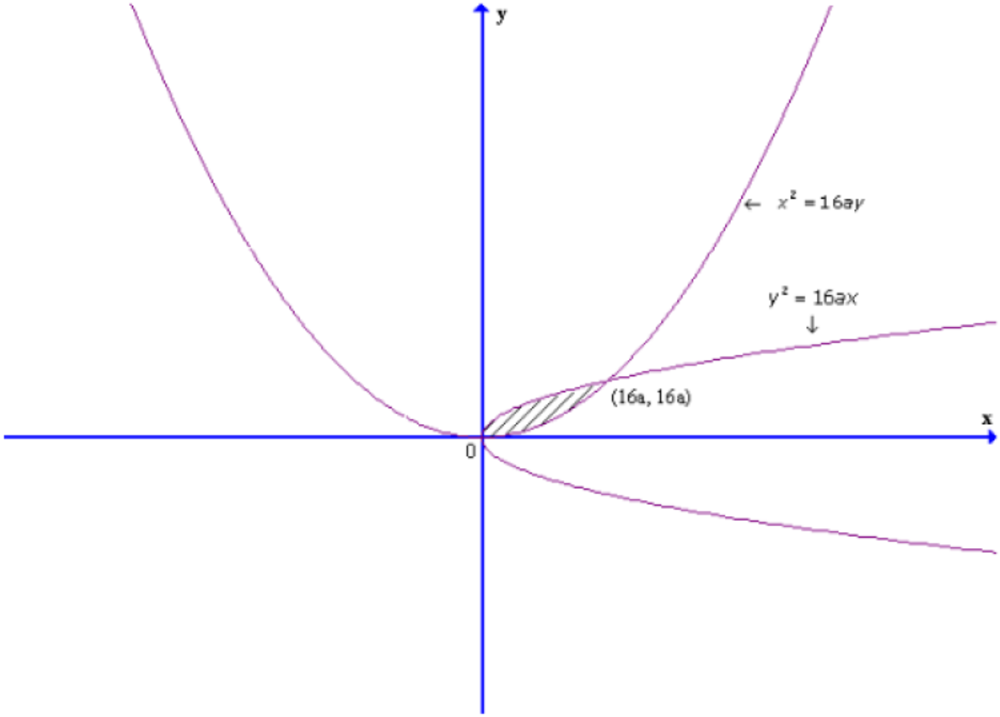
$$\frac{a^2}{12} = \int_0^a \sqrt{4ax} - mx\,dx$$

$$\frac{a^2}{12} = \left[2\sqrt{a}\,\frac{x^{3/2}}{3/2} - m\frac{x^2}{2}\right]_0^a$$

$$\frac{a^2}{12} = \frac{4a^2}{3} - m\frac{a^2}{2}$$

$m = 2$

Areas of Bounded Regions Ex-21-3 Q 51



Area of the bounded region $= \frac{1024}{3}$

$$\frac{1024}{3} = \int_0^{16a} \sqrt{16ax} - \frac{x^2}{16a}\, dx$$

$$\frac{1024}{3} = \left[4\sqrt{a}\,\frac{x^{3/2}}{3/2} - \frac{x^3}{48a}\right]_0^{16a}$$

$$\frac{1024}{3} = \frac{(16a)^2 \times 2}{3} - \frac{(16a)^3}{48a}$$

a = 2

Note: Answer given in the book is incorrect.