RD Sharma Solutions Class 11 Maths Chapter 13 Ex 13.4

Complex Numbers Ex 13.4 Q1(i)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$

where,

$|z| = \sqrt{x^2 + y^2}$ and

$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

let $z = 1 + i$

$|z| = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

$\because x, y > 0$, so θ lies in first quadrant

Now,

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$= \tan^{-1}\left(\frac{1}{1}\right) \qquad [\because a = 1 \text{ and } b = 1]$$

$$= \tan^{-1}(1)$$

$$= \tan^{-1}\left(\frac{\tan\pi}{4}\right) \qquad \left(\because \frac{\tan\pi}{4} = 1\right)$$

$$= \frac{\pi}{4} \qquad \left(\because \tan^{-1}(\tan x) = x\right)$$

$\Rightarrow \arg(z) = \frac{\pi}{4}$

Polar form of $1 + i$ is given by $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Complex Numbers Ex 13.4 Q1(ii)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$
where,

$|z| = \sqrt{x^2 + y^2}$ and

$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

let $z = \sqrt{3} + i$

$$|z| = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2}$$
$$= \sqrt{3+1}$$
$$= \sqrt{4}$$
$$= 2$$

$\because x = \sqrt{3} > 0$ & $y = 1 > 0$,
$\therefore$ θ lies in first quadrant

Hence

$$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$
$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \tan^{-1}\left(\frac{\tan\pi}{6}\right)$$
$$= \tan^{-1}\left(\because \tan^{-1}(\tan x) = x\right)$$

polar form is given by $z = |z|(\cos\theta + i\sin\theta)$

i.e $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Complex Numbers Ex 13.4 Q1(iii)

Modulus, $|1-i| = \sqrt{1^2+1^2} = \sqrt{2}$

Argument, $\arg(1-i) = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

Polar form, $\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

Complex Numbers Ex 13.4 Q1(iv)

$$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{(1-i)^2}{1^2-i^2} = \frac{1-2i-1}{1+1} = \frac{-2i}{2} = -i$$

Modulus, $\left|\frac{1-i}{1+i}\right| = |-i| = 1$

Argument, $\tan^{-1}\left(\frac{-1}{0}\right) = -\frac{\pi}{2}$

Polar Form, $z = r(\cos\theta + i\sin\theta)$

$z = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

Complex Numbers Ex 13.4 Q1(v)

$$\text{Modulus, } \left|\frac{1}{1+i}\right|$$

$$= \left|\frac{1(1-i)}{(1+i)(1-i)}\right| \text{[Rationalizing the denominator]}$$

$$= \left|\frac{1-i}{1^2-i^2}\right| = \left|\frac{1-i}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\text{Argument, } \tan^{-1}(-1) = -\frac{\pi}{4}$$

$$\text{Polar Form} = \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)$$

Complex Numbers Ex 13.4 Q1(vi)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$

where,

$|z| = \sqrt{x^2 + y^2}$ and

$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$\text{let } z = \frac{1+2i}{1-3i}$$

$$= \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= \frac{1(1+3i) + 2i(1+3i)}{1^2+3^2}$$

$$= \frac{1+3i+2i-6}{1+9}$$

$$= \frac{-5+5i}{10}$$

$$= \frac{-5}{10} + \frac{5}{10}i$$

$$= \frac{-1}{2} + \frac{1}{2}i$$

$$\therefore |z| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{2}{4}}$$

$$= \frac{1}{\sqrt{2}}$$

Here $x = \frac{-1}{2} < 0$ & $y = \frac{1}{2} > 0, \therefore \theta$ lies in quadrant II

$$\theta = \arg(z) = \tan^{-1} \frac{1/2}{-1/2}$$

$$= \tan^{-1}(-1)$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$$= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right) \qquad \left(\because \tan(\pi - \theta) = -\tan\theta\right)$$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

The polar form is given by $z = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Complex Numbers Ex 13.4 Q1(vii)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$
where,

$|z| = \sqrt{x^2 + y^2}$ and

$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

let $z = \sin 120^\circ - i\cos 120^\circ$

$$= \sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right) - i\cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right) \qquad \left(\because 120^\circ = \frac{\pi}{2} + \frac{\pi}{6}\right)$$

$$\Rightarrow z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \qquad \left(\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \ \& \ \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta\right)$$

Here z is already in polar form

with $|z| = 1$ & $\theta = \arg(z) = \frac{\pi}{6}$

Complex Numbers Ex 13.4 Q1(viii)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos\theta + i\sin\theta)$
where,

$|z| = \sqrt{x^2 + y^2}$ and

$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$

let $z = \frac{-16}{1 + i\sqrt{3}}$

$= \frac{-16}{1 + i\sqrt{3}} \times \frac{1 - i\sqrt{3}}{1 - i\sqrt{3}}$

$= \frac{-16\left(1 - i\sqrt{3}\right)}{(1)^2 + \left(\sqrt{3}\right)^2}$

$= \frac{-16\left(1 - i\sqrt{3}\right)}{1 + 3}$

$= \frac{-16}{4}\left(1 - i\sqrt{3}\right)$

$= -4\left(1 - i\sqrt{3}\right)$

$= -4 + 4\sqrt{3}i$

$\therefore |z| = \sqrt{(-4)^2 + \left(4\sqrt{3}\right)^2}$

$= \sqrt{16 + 48}$

$= \sqrt{64}$

$= 8$

Here $x = -4 < 0$ & $y = 4R3 > 0, \therefore \theta$ lies in quadrant II

$\theta = \arg(z) = \tan^{-1}\left(\frac{4\sqrt{3}}{-4}\right)$

$= \tan^{-1}\left(-\sqrt{3}\right)$

$= \tan^{-1}\left(-\tan\frac{\pi}{3}\right)$

$= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{3}\right)\right)$ $\quad \left(\because \tan(\pi - \theta) = -\tan\theta\right)$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

The polar form is given by $z = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Complex Numbers Ex 13.4 Q2

$z = (i^{25})^3 = (i)^3 = -i$

$|z| = 1,$

$\arg(z) = \tan^{-1}\left(\frac{-1}{0}\right) = -\frac{\pi}{2}$

Polar Form: $\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

Complex Numbers Ex 13.4 Q3(i)

Let $z = 1 + i\tan\alpha$

$\tan\alpha$ is periodic function with period π.

So, let us take α lying in the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Case – I : When $\alpha \in \left[0, \frac{\pi}{2}\right)$

$|z| = \sqrt{1 + \tan^2\alpha} = \sqrt{\sec^2\alpha} = |\sec\alpha| = \sec\alpha$

Let β be acute angle given by $\tan\beta = \frac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$\tan\beta = |\tan\alpha| = \tan\alpha$

$\Rightarrow \beta = \alpha$

As z is represented by a point in first quadrant.

$\therefore \arg(z) = \beta = \alpha.$

So polar form of z is $\sec\alpha(\cos\alpha + i\sin\alpha)$

Case – II : When $\alpha \in \left(\frac{\pi}{2}, \pi\right]$

$|z| = \sqrt{1 + \tan^2\alpha} = \sqrt{\sec^2\alpha} = |\sec\alpha| = -\sec\alpha$

Let β be acute angle given by $\tan\beta = \frac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$\tan\beta = |\tan\alpha| = -\tan\alpha = \tan(\pi - \alpha)$

$\Rightarrow \beta = \pi - \alpha$

As z is represented by a point in fourth quadrant.

$\therefore \arg(z) = -\beta = \alpha - \pi.$

So polar form of z is $-\sec\alpha(\cos(\alpha - \pi) + i\sin(\alpha - \pi))$.

Complex Numbers Ex 13.4 Q3(ii)

Let $z = \tan\alpha - i$

$\tan\alpha$ is periodic function with period π.

So, let us take α lying in the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Case – I : When $\alpha \in \left[0, \frac{\pi}{2}\right)$

$|z| = \sqrt{\tan^2\alpha + 1} = \sqrt{\sec^2\alpha} = |\sec\alpha| = \sec\alpha$

Let β be acute angle given by $\tan\beta = \frac{|Im(z)|}{|Re(z)|}$.

$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = \cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$

$\Rightarrow \beta = \frac{\pi}{2} - \alpha$

As z is represented by a point in fourth quadrant.

$\therefore \arg(z) = -\beta = \alpha - \frac{\pi}{2}$.

So polar form of z is $\sec\alpha\left(\cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right)\right)$

Case – II : When $\alpha \in \left(\frac{\pi}{2}, \pi\right]$

$|z| = \sqrt{\tan^2\alpha + 1} = \sqrt{\sec^2\alpha} = |\sec\alpha| = -\sec\alpha$

Let β be acute angle given by $\tan\beta = \frac{|Im(z)|}{|Re(z)|}$.

$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = -\cot\alpha = \tan\left(\alpha - \frac{\pi}{2}\right)$

$\Rightarrow \beta = \alpha - \frac{\pi}{2}$

As z is represented by a point in third quadrant.

$\therefore \arg(z) = \pi + \beta = \frac{\pi}{2} + \alpha$.

So polar form of z is $-\sec\alpha\left(\cos\left(\frac{\pi}{2} + \alpha\right) + i\sin\left(\frac{\pi}{2} + \alpha\right)\right)$.

Complex Numbers Ex 13.4 Q3(iii)

Let $z = (1 - \sin\alpha) + i\cos\alpha$

Since sine and cosine are periodic functions with period 2π

So, let us take α lying in the interval $[0, 2\pi]$.

Now, $z = (1 - \sin\alpha) + i\cos\alpha$

$$\Rightarrow |z| = \sqrt{(1 - \sin\alpha)^2 + \cos^2\alpha} = \sqrt{2 - 2\sin\alpha} = \sqrt{2}\sqrt{1 - \sin\alpha}$$

$$\Rightarrow |z| = \sqrt{2}\sqrt{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2} = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right|$$

Let β be acute angle given by $\tan\beta = \dfrac{|\text{Im}(z)|}{|\text{Re}(z)|}$.

$$\tan\beta = \frac{|\cos\alpha|}{|1 - \sin\alpha|} = \left|\frac{\cos\alpha}{1 - \sin\alpha}\right| = \left|\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2}\right| = \left|\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}\right|$$

$$\Rightarrow \tan\beta = \left|\frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}\right| = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right|$$

Following cases arise:

Case I: When $0 \le \alpha < \dfrac{\pi}{2}$

$\cos\dfrac{\alpha}{2} > \sin\dfrac{\alpha}{2}$ and $\dfrac{\pi}{4} + \dfrac{\alpha}{2} \in \left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$$\therefore |z| = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$$

and, $\tan\beta = \left|\tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)\right| = \tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right) \Rightarrow \beta = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$

Clearly, z lies in the first quadrant.

$$\therefore \arg(z) = \frac{\pi}{4} + \frac{\alpha}{2}$$

So polar form of z is $\sqrt{2}\left(\cos\dfrac{\alpha}{2} - \sin\dfrac{\alpha}{2}\right)\left(\cos\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right) + i\sin\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)\right)$

Case II: When $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$

$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2}$ and $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore\ |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right| = -\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) = \tan\left\{\pi - \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right\} = \tan\left(\frac{3\pi}{4} - \frac{\alpha}{2}\right)$

$\Rightarrow \beta = \frac{3\pi}{4} - \frac{\alpha}{2}$

Since $1 - \sin\alpha > 0$ and $\cos\alpha < 0$.

Clearly, z lies in the fourth quadrant.

$\therefore\ \arg(z) = -\beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

So polar form of z is $-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right)$

Case III: When $\frac{3\pi}{2} < \alpha < 2\pi$

$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2}$ and $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\pi, \frac{5\pi}{4}\right)$

$\therefore\ |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right| = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) = -\tan\left\{\pi - \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right\} = \tan\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)$

$\Rightarrow \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

Clearly, $Re(z) < 0$ and $Im(z) > 0$.

So, z lies in the first quadrant.

$\therefore\ \arg(z) = \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

So polar form of z is $-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right)$.

Complex Numbers Ex 13.4 Q3(iv)

Let $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\ \sin\frac{\pi}{3}} = \dfrac{1-i}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} = \dfrac{2-2i}{1+i\sqrt{3}} = \dfrac{(2-2i)(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} =$

$$\frac{(2-2\sqrt{3}) - i(2\sqrt{3}+2)}{4} = \frac{(1-\sqrt{3})}{2} - i\frac{(\sqrt{3}+1)}{2}$$

$$|z| = \sqrt{\frac{(1-\sqrt{3})^2}{4} + \frac{(\sqrt{3}+1)^2}{4}} = \sqrt{\frac{8}{4}} = \sqrt{2}$$

Let β be acute angle given by $\tan\beta = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$.

$$\tan\beta = \frac{\left|-\frac{(\sqrt{3}+1)}{2}\right|}{\left|\frac{(1-\sqrt{3})}{2}\right|} = \left|\frac{-(\sqrt{3}+1)}{(1-\sqrt{3})}\right| = |2+\sqrt{3}| = \tan\left(\frac{7\pi}{12}\right)$$

$$\Rightarrow \beta = \frac{7\pi}{12}$$

Z is represented by a point in second quadrant.

So polar form of z is $\sqrt{2}\left(\cos\dfrac{7\pi}{12} - i\sin\dfrac{7\pi}{12}\right)$.

$|z_1| = |z_2|$

Let $\arg(z_1) = \theta$

$\therefore \arg(z_2) = \pi - \theta$

In polar form, $z_1 = |z_1|(\cos\theta + i\sin\theta)$......(i)

$z_2 = |z_2|\{\cos(\pi-\theta) + i\sin(\pi-\theta)\}$

$= |z_2|(-\cos\theta + i\sin\theta)$

$= -|z_2|(\cos\theta - i\sin\theta)$

Finding conjugate of

$\overline{z_2} = -|z_2|(\cos\theta + i\sin\theta)$......(ii)

(i)/(ii) is equal to

$$\frac{z_1}{\overline{z_2}} = -\frac{|z_1|(\cos\theta + i\sin\theta)}{|z_2|(\cos\theta + i\sin\theta)}$$

$$\frac{z_1}{\overline{z_2}} = -\frac{|z_1|}{|z_1|} \qquad [\because |z_1| = |z_2|]$$

$$\frac{z_1}{\overline{z_2}} = -1$$

$z_1 = -\overline{z_2}$

Hence Proved.

Complex Numbers Ex 13.4 Q5

z_1, z_2 are conjugates implies $z_2 = \overline{z_1}$

z_3, z_4 are conjugates implies $z_4 = \overline{z_3}$

Also we know that $\arg(z_1)+\arg(\overline{z_1})=0$

$$\arg\left(\frac{z_1}{z_4}\right)+\arg\left(\frac{z_2}{z_3}\right)$$

$$=\arg(z_1)-\arg(z_4)+\arg(z_2)-\arg(z_3) \quad \left[\because \arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)\right]$$

$$=\arg(z_1)-\arg(\overline{z_3})+\arg(\overline{z_1})-\arg(z_3)$$

$$=\arg(z_1)+\arg(\overline{z_1})-\arg(\overline{z_3})-\arg(z_3)$$

$$=\arg(z_1)+\arg(\overline{z_1})-\left[\arg(\overline{z_3})+\arg(z_3)\right]\left[\because \arg(z_1)+\arg(\overline{z_1})=0\right]$$

$$= 0 + 0 = 0$$

Complex Numbers Ex 13.4 Q6

$$\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$$

$$=2\sin\frac{\pi}{10}\cos\frac{\pi}{10}+i2\sin^2\frac{\pi}{10}\left[\text{Using } \sin 2\theta=2\sin\theta\cos\theta \ \& \ 1-\cos 2\theta=2\sin^2\theta\right]$$

$$= 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$$