9. Continuity and Differentiability

Exercise 9A

1. Question

Show that ƒ(x) = x^2 is continues at x=2.

Answer

Left Hand Limit: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2$

= 4

Right Hand Limit: $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2$

= 4

ƒ(2) = 4

Since, $\lim_{x\to 2} f(x)$ = f(2)

∴ f is continuous at x=2.

2. Question

Show that ƒ(x) = (x^2+3x+4) is continuous at x=1.

Answer

Left Hand Limit: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 + 3x + 4$

= 7

Right Hand Limit: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 + 3x + 4$

= 7

ƒ(1) = 7

Since, $\lim_{x\to 1} f(x)$ = f(1)

∴ f is continuous at x=1.

3. Question

Prove that

ƒ(x) = $\begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{when } x \neq 3 \\ 5, & \text{when } x = 3 \end{cases}$ is continuous at x=3

Answer

LHL: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-x-6}{x-3}$

$= \lim_{x\to 3^-} \frac{(x+2)(x-3)}{x-3}$ [By middle term splitting]

$= \lim_{x\to 3^-} x + 2$

= 5

RHL: $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2-x-6}{x-3}$

$= \lim_{x \to 3^+} \frac{(x+2)(x-3)}{x-3}$ [By middle term splitting]

$= \lim_{x \to 3^+} x + 2$

= 5

f(3) = 5

Since, $\lim_{x \to 3} f(x)$ = f(3)

∴ f is continuous at x=3.

4. Question

Prove that

$f(x) = \begin{cases} \frac{x^2 - 25}{x - 5}, & \text{when } x \neq 5 \\ 10, & \text{when } x = 5 \end{cases}$ is continuous at x=5

Answer

LHL: $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} \frac{x^2-25}{x-5}$

$= \lim_{x \to 5^-} \frac{(x+5)(x-5)}{x-5}$ [By middle term splitting]

$= \lim_{x \to 5^-} x + 5$

= 10

RHL: $\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} \frac{x^2-25}{x-5}$

$= \lim_{x \to 5^+} \frac{(x+5)(x-5)}{x-5}$ [By middle term splitting]

$= \lim_{x \to 5^+} x + 5$

= 10

f(5)= 10

Since, $\lim_{x \to 5} f(x)$ = f(5)

∴ f is continuous at x=5.

5. Question

Prove that

$f(x) = \begin{cases} \frac{\sin 3x}{x}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases}$ is discontinuous at x=0

Answer

LHL: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin 3x}{x}$

= 3

$[\lim_{x \to a} \frac{\sin nx}{x}$ = n]

RHL: $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{\sin 3x}{x}$

= 3

f(0)=1

Since, $\lim\limits_{x\to 0} f(x) \neq$ f(0)

∴ f is discontinuous at x=0.

6. Question

Prove that

$f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & \text{when } x \neq 0; \\ 1, & \text{when } x = 0 \end{cases}$ is discontinuous at x=0

Answer

LHL: $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{1-\cos x}{x^2}$

$= \lim\limits_{x\to 0^-} \frac{2\sin^2\frac{x}{2}}{x^2}$

$= 2 \lim\limits_{x\to 0^-} \frac{(\sin\frac{x}{2})^2}{x^2}$

$= 2 \times \frac{1}{4}$

$= \frac{1}{2}$

RHL: $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{1-\cos x}{x^2}$

$= \lim\limits_{x\to 0^+} \frac{2\sin^2\frac{x}{2}}{x^2}$

$= 2 \lim\limits_{x\to 0^+} \frac{(\sin\frac{x}{2})^2}{x^2}$

$= 2 \times \frac{1}{4}$

$= \frac{1}{2}$

f(0)= 1

Since, $\lim\limits_{x\to 0} f(x) \neq$ f(0)

∴ f is discontinuous at x=0.

7. Question

Prove that

$f(x) = \begin{cases} 2-x, & \text{when } x < 2; \\ 2+x, & \text{when } x \geq 2 \end{cases}$ is discontinuous at x=2

Answer

LHL: $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} 2 + x$

= 4

RHL: $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2 - x$

= 0

$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

∴ f(x) is discontinuous at x=2

8. Question

Prove that

f(x) = $\begin{cases} 3 - x, & \text{when } x \le 0; \\ x^2, & \text{when } x > 0 \end{cases}$ is discontinuous at x=0

Answer

LHL: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3 - x$

= 3

RHL: $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2$

= 0

$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

∴ f(x) is discontinuous at x=0

9. Question

Prove that

f(x) = $\begin{cases} 5x - 4, & \text{when } 0 < x \le 1; \\ 4x^2 - 3x, & \text{when } 1 < x < 2 \end{cases}$ is continuous at x=1

Answer

LHL: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 5x - 4$

= 1

RHL: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4x^2 - 3x$

= 1

f(x)=5x-4 [this equation is taken as equality for x=1 lies there]

f(1)= 1

Since, $\lim_{x\to 1} f(x)$ = f(1)

∴ f is continuous at x=1.

10. Question

Prove that

f(x) = $\begin{cases} x - 1, & \text{when } 1 \le x < 2; \\ 2x - 3, & \text{when } 2 \le x \le 3 \end{cases}$ is continuous at x=2

Answer

LHL: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x - 1$

= 1

RHL: $\lim_{x\to2^+} f(x) = \lim_{x\to2^+} 2x-3$

= 1

f(x)=2x-3 [this equation is taken as equality for x=1 lies there]

f(2)= 1

Since, $\lim_{x\to2} f(x)$ = f(2)

∴ f is continuous at x=2.

11. Question

Prove that

f(x) = $\begin{cases} \cos x, & \text{when } x \ge 0; \\ -\cos x, & \text{when } x < 0 \end{cases}$ is discontinuous at x=0

Answer

LHL: $\lim_{x\to0^-} f(x) = \lim_{x\to0^-} \cos x$

= 1

RHL: $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} -\cos x$

= -1

$\lim_{x\to0^-} f(x) \ne \lim_{x\to0^+} f(x)$

∴ f(x) is discontinuous at x=0

12. Question

Prove that

f(x) = $\begin{cases} \frac{|x-a|}{x-a}, & \text{when } x \ne a; \\ 1, & \text{when } x = a \end{cases}$ is discontinuous at x=a

Answer

LHL: $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} \frac{|x-a|}{x-a}$

$= \lim_{x\to a^-} \frac{-(x-a)}{x-a}$

= -1

RHL: $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} \frac{|x-a|}{x-a}$

$= \lim_{x\to a^+} \frac{(x-a)}{x-a}$

= 1

$\lim_{x\to a^-} f(x) \ne \lim_{x\to a^+} f(x)$

∴ f(x) is discontinuous at x=a

13. Question

Prove that

f(x) = $\begin{cases} \frac{1}{2}(x-|x|), & \text{when } x \neq 0; \\ 2, & \text{when } x = 0 \end{cases}$ is discontinuous at x=0

Answer

LHL: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{2}(x-|x|)$

$= \lim_{x\to 0^-} \frac{1}{2}(x-(-x))$

$= \lim_{x\to 0^-} 2x$

= 0

RHL: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{2}(x-|x|)$

$= \lim_{x\to 0^-} \frac{1}{2}(x-(x))$

= 0

f(0)=2

Since, $\lim_{x\to 0} f(x) \neq$ f(0)

∴ f is discontinuous at x=0.

14. Question

Prove that

f(x) = $\begin{cases} \sin\frac{1}{x}, & \text{when } x \neq 0; \\ 0, & \text{when } x = 0; \end{cases}$ is discontinuous at x=0

Answer

$\lim_{x\to 0} \sin\frac{1}{x}$ = 0

$\sin\frac{1}{x}$ is bounded function between -1 and +1.

Also, f(0)=0

Since, $\lim_{x\to 0} f(x)$ = f(0)

Hence, f is a continuous function.

15. Question

Prove that

f(x) = $\begin{cases} 2x, & \text{when } x < 2; \\ 2, & \text{when } x = 2; \\ x^2, & \text{when } x > 2; \end{cases}$ is discontinuous at x=2

Answer

LHL: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x$

=4

RHL: $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2$

=4

f(2)=2

Since, $\lim_{x \to 0} f(x) \neq$ f(2)

∴ f is discontinuous at x=2.

16. Question

Prove that

f(x) = $\begin{cases} -x, & \text{when } x < 0; \\ 1, & \text{when } x = 0; \\ x, & \text{when } x > 0; \end{cases}$ is discontinuous at x=0

Answer

LHL: $\lim_{x \to 0^-} f(x)$ = -x

=0

RHL: $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x$

=0

f(0)=1

Since, $\lim_{x \to 0} f(x) \neq$ f(0)

∴ f is discontinuous at x=0.

17. Question

Find the value of k for which

f(x) = $\begin{cases} \dfrac{\sin 2x}{5x}, & \text{when } x \neq 0; \\ \lambda, & \text{when } x = 0 \end{cases}$ is continuous at x=0

Answer

Since, f(x) is continuous at x=0

$\Rightarrow \lim_{x \to 0} \frac{\sin 2x}{5x}$ = f(0)

$\Rightarrow \frac{1}{5} \lim_{x \to 0} \frac{\sin 2x}{x} = \lambda$

$\Rightarrow \frac{1}{5}$ x 2 = λ

$\Rightarrow \lambda = \frac{2}{5}$

18. Question

Find the value of λ for which

f(x) = $\begin{cases} \dfrac{x^2 - 2x - 3}{x + 1}, & \text{when } x \neq -1; \\ \lambda, & \text{when } x = -1 \end{cases}$ is continuous at x=-1

Answer

Since, f(x) is continuous at x=0

$\Rightarrow \lim_{x\to-1}\frac{x^2-2x-3}{x+1} = f(0)$

$\Rightarrow \lim_{x\to-1}\frac{(x-3)(x+1)}{x+1} = \lambda$

$\Rightarrow \lim_{x\to-1} x-3 = \lambda$

$\Rightarrow \lambda = -4$

19. Question

For what valve of k is the following function continuous at x=2

$$f(x) = \begin{cases} 2x+1, & \text{when } x<2 \\ k, & \text{when } x=2 \\ 3x-1, & \text{when } x>2 \end{cases}$$

Answer

Since, f(x) is continuous at x=2

$\Rightarrow \lim_{x\to2^-} 2x+1 = \lim_{x\to2^+} 3x-1 = f(2)$

$\Rightarrow \lim_{x\to2^-} 2x+1 = f(2)$

$\Rightarrow k = 5$

20. Question

For what valve of k is the following function

$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{when } x\neq3; \\ k, & \text{when } x=3 \end{cases}$ is continuous at x=3

Ans. k=6

Answer

Since, f(x) is continuous at x=3

$\Rightarrow \lim_{x\to3}\frac{x^2-9}{x-3} = f(3)$

$\Rightarrow \lim_{x\to3}\frac{(x-3)(x+3)}{x-3} = f(3)$

$\Rightarrow \lim_{x\to3}(x+3) = f(3)$

$\Rightarrow k = 9$

21. Question

For what valve of k is the following function

$f(x) = \begin{cases} \frac{k\cos x}{\pi-2x}, & \text{if } x\neq\frac{\pi}{2}; \\ 3, & \text{if } x=\frac{\pi}{2} \end{cases}$ is continuous at $x=\frac{\pi}{2}$

Ans. k=6

Answer

f is continuous at x = $\frac{\pi}{2}$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} f(x)$= f($\frac{\pi}{2}$)

$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{k\cos x}{\pi-2x}$= 3

$\Rightarrow \lim_{h\to 0} \frac{k\cos(\frac{\pi}{2}-h)}{\pi-2(\frac{\pi}{2}-h)}$= 3 [Here x = $\frac{\pi}{2}$ - h]

$\Rightarrow \lim_{h\to 0} \frac{k\sin h}{\pi-\pi+2h}$= 3

$\Rightarrow \lim_{h\to 0} \frac{k\sin h}{2h}$= 3

$\Rightarrow \frac{k}{2}$ x 1 = 3

$\Rightarrow$ k = 6

22. Question

Show that function:

ƒ(x) = $\begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$ is continuous at x=0

Answer

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\sin\frac{1}{x}$

As $\lim_{x\to 0} x^2$ = 0 and sin($\frac{1}{x}$) is bounded function between -1 and +1.

$\therefore \lim_{x\to 0} x^2\sin\frac{1}{x}$ = 0

Also, f(0)=0

Since, $\lim_{x\to 0} f(x)$ = f(0)

Hence, f is a continuous function.

23. Question

Show that: ƒ(x) = $\begin{cases} x+1, & \text{if } x \geq 1; \\ x^2+1, & \text{if } x < 1 \end{cases}$ is continuous at x=1

Answer

: LHL: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 + 1$

= 2

RHL: $\lim_{x\to 2^+} f(x) = \lim_{x\to 1^+} x + 1$

= 2

f(1)= 2

Since, $\lim_{x \to 1} f(x)$ = f(1)

∴ f is continuous at x=1.

24. Question

Show that: f(x) = $\begin{cases} x^3 - 3, \text{ if } x \le 2; \\ x^2 + 1, \text{ if } x > 2 \end{cases}$ is continuous at x=2

Answer

: LHL: $\lim_{x \to 2^-} f(x) = \lim_{x \to 1^-} x^3 - 3$

= 5

RHL: $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 + 1$

= 5

f(2)= 5

Since, $\lim_{x \to 2} f(x)$ = f(2)

∴ f is continuous at x=2.

25. Question

Find the values of a and b such that the following functions continuous. $\begin{cases} 5, \text{ when } x \le \\ ax + b, \text{ when } 2 < x < 10 \\ 21, \text{ when } x \ge 10 \end{cases}$

Answer

f is continuous at x=2

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2+} f(x)$ = f(2)

$\lim_{x \to 2^-} (5) = \lim_{x \to 2+} [ax + b]$ = 5

⇒ 2a+b=5 ……. (1)

f is continuous at x=10

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2+} f(x)$ = f(2)

$\lim_{x \to 2^-} (21) = \lim_{x \to 2+} [ax + b]$ = 21

⇒ 10a+b=21 ……. (1)

(1) - (2)

-8a = -16

a = 2

Putting a in 1

b=1

26. Question

Find the values of a and b such that the following functions ƒ, defined as $\begin{cases} a\sin\frac{\pi}{2}(x+1), & x \le 0; \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$ is continuous at x=0

Answer

: f is continuous at x=0

$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x)$

$\lim_{x\to 0-}(a\sin\frac{\pi}{2}(x+1)) = \lim_{x\to 0+}[\frac{\tan x - \sin x}{x^3}]$

$(a\sin\frac{\pi}{2}(0+1) = \lim_{x\to 0+}[\frac{\frac{\sin x}{\cos x} - \sin x}{x^3}]$

a = $\lim_{x\to 0+}[\frac{\sin x(\frac{1}{\cos x}-1)}{x^3}]$

= $\lim_{x\to 0+}[\frac{\sin x(\frac{1}{\cos x}-1)}{x^3}]$

= $\lim_{x\to 0+}[\frac{\sin x(1-\cos x)}{\cos x\, x^3}]$

= $\lim_{x\to 0+}[\frac{\sin x\, .2\sin^2\frac{x}{2}}{\cos x\, x^3}]$

= $\lim_{x\to 0+}[\frac{\sin x\, .2\sin^2\frac{x}{2}}{x\, .x^2}]$ x $\frac{1}{\cos x}$

= 1 x 2 x$\frac{1}{4}$ x 1

= $\frac{1}{2}$

27. Question

Prove that the function ƒ given ƒ(x)=|x-3|, x ϵ R is continuous but not differentiable at x=3

Answer

f(x)=|x-3|

Since every modulus function is continuous for all real x, f(x) is continuous at x=3.

f(x) = $f(x) = \begin{cases} 3 - x, x < 0 \\ x - 3, x \ge 0 \end{cases}$

To prove differentiable , we will use the following formula.

$\lim_{x\to a+}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a-}\frac{f(x)-f(a)}{x-a}$ = f(a)

L.H.L $\lim_{x\to a+}\frac{f(x)-f(a)}{x-a}$

= $\lim_{x\to 3+}\frac{x-3-0}{x-3}$

= $\lim_{x\to 3-}\frac{x-3}{x-3}$

= 1

R.H.L: $\lim_{x\to a-}\frac{f(x)-f(a)}{x-a}$

$= \lim_{x \to 3^-} \frac{3-x-0}{x-3}$

$= \lim_{x \to 3^-} \frac{3-x}{x-3}$

= -1

Since, L.H.L $\neq$ R.H.L, f(x) is not differentiable at x=5.

Exercise 9B

1. Question

Show that function ƒ(x) = $\begin{cases} (7x+5), & \text{when } x \geq 0; \\ (5-3x), & \text{when } x < 0 \end{cases}$ is continuous function.

Answer

Given:

ƒ(x) = $\begin{cases} (7x+5), \text{when} x \geq 0; \\ (5-3x), \text{when} x < 0 \end{cases}$

Let's calculate the limit of f(x) when x approaches 0 from the right

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (7x+5)$ = 7(0) + 5

= 5

Therefore,

$\lim_{x \to 0^+} f(x) = 5$

Let's calculate the limit of f(x) when x approaches 0 from the left

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (5-3x)$ = 5 – 3(0)

= 5

Therefore,

$\lim_{x \to 0^-} f(x) = 5$

Also, f(0) = 5

As we can see,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ = f(0) = 5

Thus, we can say that f(x) is continuous function.

2. Question

Show that function ƒ(x) = $\begin{cases} \sin x, & \text{if } x < 0; \\ x, & \text{if } x \geq 0 \end{cases}$ is continuous.

Answer

Given:

ƒ(x) = $\begin{cases} \sin x, \text{if } x < 0; \\ x, \text{if} x \geq 0 \end{cases}$

Left hand limit at x = 0

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (\sin x) = \sin(0) = 0$

Therefore,

$\lim_{x \to 0^-} f(x) = 0$

Right hand limit at x = 0

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = 0$

Therefore,

$\lim_{x \to 0^+} f(x) = 0$

Also, f(0) = 0

As,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$

Thus, we can say that f(x) is continuous function.

3. Question

Show that function $f(x) = \begin{cases} \frac{x^n - 1}{x - 1}, & \text{when } x \neq 1; \\ n, & \text{when } x = 1 \end{cases}$ is continuous.

Answer

Given:

$f(x) = \begin{cases} \frac{x^n - 1}{x - 1}, \text{when} x \neq 1; \\ n, \text{when} x = 1 \end{cases}$

Left hand limit and x = 1

$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} \frac{(1-h)^n - 1}{(1-h) - 1}$

$\lim_{h \to 0} \frac{(1-h)^n - 1}{1 - h - 1} = \lim_{h \to 0} \frac{(1-h)^n - 1}{-h} = \lim_{h \to 0} - \frac{(1-h)^n - 1}{h}$

$= -\lim_{h \to 0} \frac{(1-h)^n - 1}{h}$ (Because $\lim_{x \to a} c.f(x) = c \lim_{x \to a} f(x)$)

Applying L hospital's rule $\left(\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}\right)$

$= -\lim_{h \to 0} \frac{-n(1-h)^{n-1}}{1} = -[-n(1-0)^{n-1}] = n$

Right hand limit and x = 1

$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1 + h) = \lim_{h \to 0} \frac{(1+h)^n - 1}{(1+h) - 1}$

$\lim_{h \to 0} \frac{(1+h)^n - 1}{1 + h - 1} = \lim_{h \to 0} \frac{(1+h)^n - 1}{h}$

Applying L hospital's rule $\left(\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}\right)$

$= \lim_{h \to 0} \frac{n(1+h)^{n-1}}{1} = [n(1+0)^{n-1}] = n$

Also, f(x) = n at x =1

As we can see that $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(x)$

Thus, f(x) is continuous at x = 1

4. Question

Show that sec x is a continuous function.

Answer

Let f(x) = sec x

Therefore, f(x) = $\frac{1}{\cos x}$

f(x) is not defined when cos x = 0

And cos x = 0 when, x = $\frac{\pi}{2}$ and odd multiples of $\frac{\pi}{2}$ like $-\frac{\pi}{2}$

Let us consider the function

f(a) = cos a and let c be any real number. Then,

$\lim_{a \to c^+} f(a) = \lim_{h \to 0} f(c+h)$

$\lim_{h \to 0} \cos(c+h) = \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$

= cos c $\lim_{h \to 0} \cos h$ - sin c $\lim_{h \to 0} \sin h$

= cos c (1) – sin c (0)

Therefore,

$\lim_{a \to c^+} f(a) =$ cos c

Similarly,

$\lim_{a \to c^-} f(a) =$ f(c) = cos c

Therefore,

$\lim_{a \to c^-} f(a) = \lim_{a \to c^+} f(a) =$ f(c) = cos c

So, f(a) is continuous at a = c

Similarly, cos x is also continuous everywhere

Therefore, sec x is continuous on the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

5. Question

Show that sec |x| is a continuous function

Answer

Let f(x) = sec |x| and a be any real number. Then,

Left hand limit at x = a

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} \sec|x| = \lim_{h \to 0} \sec|a-h| = \sec|a|$

Right hand limit at x = a

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} \sec|x| = \lim_{h \to 0} \sec|a+h| = \sec|a|$

Also, f(a) = sec |a|

Therefore,

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) =$ f(a)

Thus, f(x) is continuous at x = a.

6. Question

Show that function ƒ(x) = $\begin{cases} (2-x), & \text{when } x \geq 1; \\ x, & \text{when } 0 \leq x \leq 1. \end{cases}$ is continuous.

Answer

We know that sin x is continuous everywhere

Consider the point x = 0

Left hand limit:

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(\frac{\sin x}{x}\right) = \lim_{h \to 0} \left(\frac{\sin(0-h)}{0-h}\right) = \lim_{h \to 0} \left(\frac{-\sin h}{-h}\right)$ = 1

Right hand limit:

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right) = \lim_{h \to 0} \left(\frac{\sin(0+h)}{0+h}\right) = \lim_{h \to 0} \left(\frac{\sin h}{h}\right)$ = 1

Also we have,

f(0) = 2

As,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq$ f(0)

Therefore, f(x) is discontinuous at x = 0.

7. Question

Discuss the continuity of f(x) = [x].

Answer

Let n be any integer

[x] = Greatest integer less than or equal to x.

Some values of [x] for specific values of x

[3] = 3

[4.4] = 4

[-1.6] = -2

Therefore,

Left hand limit at x = n

$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [x]$ = n – 1

Right hand limit at x = n

$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [x]$ = n

Also, f(n) = [n] = n

As $\lim_{x \to n^-} f(x) \neq \lim_{x \to n^+} f(x)$

Therefore, f(x) = [x] is discontinuous at x = n.

8. Question

Show that ƒ(x) = $\begin{cases} (2x-1), \text{ if } x < 2; \\ \frac{3x}{2}, \text{ if } x \geq 2 \end{cases}$ is continuous.

Answer

Given function f(x) = $\begin{cases} (2x-1), \text{if } x < 2; \\ \frac{3x}{2}, \text{if } x \geq 2 \end{cases}$

Left hand limit at x = 2

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} (2x-1)$ = 2(2) - 1 = 3

Right hand limit at x = 2

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} \frac{3x}{2} = \frac{3(2)}{2}$ = 3

Also,

f(2) = $\frac{3(2)}{2}$ = 3

As

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$ = 3

Therefore,

The function f(x) is continuous at x = 2.

9. Question

Show that ƒ(x) = $\begin{cases} x, \text{ if } x \neq 0; \\ 1, \text{ if } x = 0 \end{cases}$ is continuous at each point except 0.

Answer

Given function is ƒ(x) = $\begin{cases} x, \text{if } x \neq 0; \\ 1, \text{if } x = 0 \end{cases}$

Left hand limit at x = 0

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$ = 0

Right hand limit at x = 0

$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$ = 0

Also,

f(0) = 1

As,

$\text{Lim}_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$

f(x) = x for other values of x expect 0 f(x) = 1,2,3,4…

Therefore,

f(x) is not continuous everywhere expect at x = 0

10. Question

Locate the point of discontinuity of the function

f(x) = $\begin{cases} \left(x^3 - x^2 + 2x - 2\right), & \text{if } x \neq 1; \\ 4, & \text{if } x = 0 \end{cases}$

Answer

Given function f(x) = $\begin{cases} (x^3 - x^2 + 2x - 2), \text{if } x \neq 1; \\ 4, \text{if } x = 1 \end{cases}$

Left hand limit at x = 1: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^3 - x^2 + 2x - 2)$

$= \lim_{h \to 0} \{(1-h)^3 - (1-h)^2 + 2(1-h) - 2\}$

$= \lim_{h \to 0} (1-h)^3 - \lim_{h \to 0} (1-h)^2 + 2 \lim_{h \to 0} (1-h) - 2$

= 1 – 1 + 2 – 2

= 0

Right hand limit at x = 1: $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^3 - x^2 + 2x - 2)$

$= \lim_{h \to 0} \{(1+h)^3 - (1+h)^2 + 2(1+h) - 2\}$

$= \lim_{h \to 0} (1+h)^3 - \lim_{h \to 0} (1+h)^2 + 2 \lim_{h \to 0} (1+h) - 2$

= 1 – 1 + 2 – 2

= 0

Also, f(1) = 4

As we can see that,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \neq f(1)$

Therefore,

f(x) is not continuous at x = 1

11. Question

Discus the continuity of the function ƒ(x)= |x|+|x-1| in the interval of [-1, 2]

Answer

Given function f(x) = |x| + |x - 1|

A function f(x) is said to be continuous on a closed interval [a, b] if and only if,

(i) f is continuous on the open interval (a, b)

(ii) $\lim_{x \to a^+} f(x) = f(a)$

(iii) $\lim_{x \to b^-} f(x) = f(b)$

Let’s check continuity on the open interval (-1, 2)

As -1 < x < 2

Left hand limit:

$\lim_{x \to -1^-} f(x) = \lim_{h \to 0} \{|-1-h| + |(-1-h) - 1|\}$

=|-1-0| + |(-1-0) – 1|

=1 + 2

= 3

Right hand limit:

$\lim_{x\to 2^+} f(x) = \lim_{h\to 0}\{|2+h| + |(2+h)-1|\}$

=|2| + |2 – 1|

=2+1

= 3

Left hand limit = Right hand limit

Here a = -1 and b = 2

Therefore,

$\lim_{x\to -1^+} f(x) = \lim_{h\to 0}\{|-1+h| + |(-1+h)-1|\}$

= |-1 + 0| + |(-1 + 0) - 1|

= |- 1| + |-1 – 1|

= 1 + 2 = 3

Also f(-1) = |-1| + |-1 - 1| = 1 + 2 = 3

Now,

$\lim_{x\to 2^-} f(x) = \lim_{h\to 0}\{|2-h| + |(2-h)-1|\}$

= |2 - 0| + |(2 - 0) - 1|

= | 2| + |2 – 1|

= 2 + 1 = 3

Also f(2) = |2| + |2 - 1| = 2 + 1 = 3

Therefore,

f(x) is continuous on the closed interval [-1, 2].

Exercise 9C

1. Question

Show that ƒ(x) = x^3 is continuous as well as differentiable at x=3.

Answer

Given:

f(x) = x^3

If a function is differentiable at a point, it is necessarily continuous at that point.

Left hand derivative (LHD) at x = 3

$\lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = \lim_{h\to 0} \frac{f(3-h)-f(3)}{(3-h)-3}$

$= \lim_{h\to 0} \frac{(3-h)^3 - 3^3}{(3-h)-3} = \lim_{h\to 0} \frac{(3-h)^3 - 27}{-h} = \lim_{h\to 0} - \frac{h\{(3-h)^2+3(3-h)+9\}}{h}$

$= \lim_{h\to 0} - \{(3-h)^2 + 3(3-h) + 9\} = \lim_{h\to 0} - [-\{-(3-h)^2 - 3(3-h) - 9\}]$

$= \lim_{h \to 0} -\{-h^2 + 9h - 27\} = \lim_{h \to 0} h^2 - 9h + 27 = 0^2 - 9(0) + 27 = 27$

Right hand derivative (RHD) at x = 3

$\lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{h \to 0} \frac{f(3+h)-f(3)}{(3+h)-3}$

$= \lim_{h \to 0} \frac{(3+h)^3 - 3^3}{(3+h)-3} = \lim_{h \to 0} \frac{(3+h)^3 - 27}{h} = \lim_{h \to 0} \frac{h\{(3+h)^2+3(3+h)+9\}}{h}$

$= \lim_{h \to 0} \{(3+h)^2 + 3(3+h) + 9\} = \lim_{h \to 0} (3+h)^2 + 3(3+h) + 9$

$= \lim_{h \to 0} \{h^2 + 9h + 27\} = 0^2 + 9(0) + 27 = 27$

LHD = RHD

Therefore, f(x) is differentiable at x = 3.

$\lim_{x \to 3} f(x) = \lim_{x \to 3} x^3 = 3^3 = 27$

Also, f(3) =27

Therefore, f(x) is also continuous at x = 3.

2. Question

Show that ƒ(x) = $(x-1)^{1/3}$ is not differentiable at x=1.

Answer

Given function f(x) = $(x-1)^{1/3}$

LHD at x = 1

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{h \to 0} \frac{f(1-h)-f(1)}{(1-h)-1} = \lim_{h \to 0} \frac{\{(1-h)-1\}^{\frac{1}{3}}(1-1)^{\frac{1}{3}}}{(1-h)-1}$

$= \lim_{h \to 0} \frac{(-h)^{\frac{1}{3}}(0)^{\frac{1}{3}}}{-h} = \frac{0}{0}$ = Not defined

RHD at x = 1

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{h \to 0} \frac{f(1+h)-f(1)}{(1+h)-1} = \lim_{h \to 0} \frac{\{(1+h)-1\}^{\frac{1}{3}}(1-1)^{\frac{1}{3}}}{(1+h)-1}$

$= \lim_{h \to 0} \frac{(-h)^{\frac{1}{3}}(0)^{\frac{1}{3}}}{-h} = \frac{0}{0}$ = Not defined

Since, LHD and RHD doesn’t exists

Therefore, f(x) is not differentiable at x = 1.

3. Question

Show that constant function is always differentiable

Answer

Let a be any constant number.

Then, f(x) = a

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

We know that coefficient of a linear function is

$a = \frac{y_2-y_1}{x_2-x_1}$

Since our function is constant, $y_1 = y_2$

Therefore, a = 0

Now,

$$f'(x) = \lim_{h\to 0}\frac{a-a}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0$$

Thus, the derivative of a constant function is always 0.

4. Question

Show that ƒ(x) = |x-5| is continuous but not differentiable at x=5

Answer

Left hand limit at x = 5

$$\lim_{x\to 5^-}|x-5| = \lim_{x\to 5}(5-x) = 0$$

Right hand limit at x = 5

$$\lim_{x\to 5^+}|x-5| = \lim_{x\to 5}(x-5) = 0$$

Also f(5) = |5 – 5 |= 0

As,

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$$

Therefore, f(x) is continuous at x = 5

Now, lets see the differentiability of f(x)

LHD at x = 5

$$\lim_{x\to 5^-}\frac{f(x)-f(5)}{x-5} = \lim_{h\to 0}\frac{f(5-h)-f(5)}{5-h-5} = \lim_{h\to 0}\frac{|5-(5-h)|-|5-5|}{-h} = \lim_{h\to 0} -\frac{h}{h} = -1$$

RHD at x = 5

$$\lim_{x\to 5^+}\frac{f(x)-f(5)}{x-5} = \lim_{h\to 0}\frac{f(5+h)-f(5)}{5+h-5} = \lim_{h\to 0}\frac{|(5+h)-5|-|5-5|}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

Since, LHD ≠ RHD

Therefore,

f(x) is not differentiable at x = 5

5. Question

Let ƒ(x) = $\begin{cases}(2-x), & \text{when } x \geq 1; \\ x, & \text{when } 0 \leq x \leq 1.\end{cases}$

Show that ƒ(x) is continuous but not differentiable at x=1

Answer

Left hand limit at x = 1

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x = 1$$

f(x) = x is polynomial function and a polynomial function is continuous everywhere

Right hand limit at x = 1

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(2-x) = (2-1) = 1$$

f(x) = 2 - x is polynomial function and a polynomial function is continuous everywhere

Also, f(1) =1

As we can see that,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

Therefore,

f(x) is continuous at x =1

Now,

LHD at x = 1

$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x-1}{x-1} = \lim_{x\to 1} \frac{1}{1} = \lim_{x\to 1} 1 = 1$

RHD at x = 1

$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{2-x-(2-1)}{x-1} = \lim_{x\to 1} \frac{2-x-1}{x-1} = \lim_{x\to 1} \frac{-(x-1)}{x-1}$

$\lim_{x\to 1} -\frac{1}{1} = \lim_{x\to 1} -1 = -1$

As, LHD ≠ RHD

Therefore,

f(x) is not differentiable at x = 1

6. Question

Show that ƒ(x) = [x] is neither continuous nor derivable at x=2.

Answer

Left hand limit at x = 2

$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} [2-h] = \lim_{h\to 0} 1 = 1$

Right hand limit at x = 2

$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [2+h] = \lim_{h\to 0} 2 = 2$

As left hand limit ≠ right hand limit

Therefore, f(x) is not continuous at x = 2

Lets see the differentiability of f(x):

LHD at x = 2

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{h\to 0} \frac{f(x-h)-f(2)}{(x-h)-2} = \lim_{h\to 0} \frac{f(2-h)-f(2)}{(2-h)-2}$$
$$= \lim_{h\to 0} -\frac{1-2}{h}$$

$\lim_{h\to 0} -\frac{(-1)}{h} = \infty$

RHD at x = 2

$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{h\to 0} \frac{f(x+h)-f(2)}{(x+h)-2} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{(2+h)-2} = \lim_{h\to 0} \frac{2-2}{h}$

$\lim_{h\to 0} \frac{0}{h} = 0$

As, LHD ≠ RHD

Therefore,

f(x) is not derivable at x = 2

7. Question

Show that function

f(x) = $\begin{cases} (1-x), & \text{when } x < 1; \\ (x^2-1), & \text{when } x \geq 1. \end{cases}$ is continuous but not differentiable at x=1

Answer

Given function f(x) = $\begin{cases} (1-x), \text{when} x < 1; \\ (x^2-1), \text{when} x \geq 1. \end{cases}$

Left hand limit at x = 1:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (1-x) = 1 - 1 = 0$$

Right hand limit at x = 1:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (x^2 - 1) = 1^2 - 1 = 0$$

Also, f(1) = 1^2 – 1 = 0

As,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

Therefore,

f(x) is continuous at x = 1

Now, let's see the differentiability of f(x):

LHD at x = 2:

$$\lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2} \frac{(1-x)-(1-2)}{x-2} = \lim_{x \to 2} \frac{1-x-1+2}{x-2} = \lim_{x \to 2} \frac{-(x-2)}{x-2}$$

$$= \lim_{x \to 2} -1 = -1$$

RHD at x = 2:

$$\lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2} \frac{(x^2-1)-(2^2-1)}{x-2} = \lim_{x \to 2} \frac{x^2-1-3}{x-2} = \lim_{x \to 2} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2} \frac{x^2-2^2}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2} (x+2) = 2 + 2 = 4$$

As, LHD ≠ RHD

Therefore,

f(x) is not differentiable at x = 2

8. Question

Let f(x) = $\begin{cases} (2+x), & \text{if } x \geq 0; \\ (2-x), & \text{if } x < 0. \end{cases}$ Show that f(x) is not derivable at x=0.

Answer

Given function f(x) = $\begin{cases} (2+x), \text{if} x \geq 0; \\ (2-x), \text{if} x < 0. \end{cases}$

LHD at x = 0:

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{(2-x)-(2)}{x-0} = \lim_{x\to 0}\frac{-x}{x}$$

$$= \lim_{x\to 0} -1 = -1$$

RHD at x = 0:

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{(2+x)-(2)}{x-0} = \lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1$$

As, LHD ≠ RHD

Therefore,

f(x) is not differentiable at x = 0

9. Question

If ƒ(x) = |x| show that ƒ'(2)=1

Answer

Given function is f(x) = |x|

LHD at x = 2:

$$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{2-h-2} = \lim_{h\to 0}\frac{|2-h|-|2|}{-h} = \lim_{h\to 0}\frac{-h}{-h}$$

$$\lim_{h\to 0} 1 = 1$$

RHD at x = 2:

$$\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{2+h-2} = \lim_{h\to 0}\frac{|2+h|-|2|}{h} = \lim_{h\to 0}\frac{h}{h}$$

$$\lim_{h\to 0} 1 = 1$$

As, LHD = RHD

Therefore, f(x) = |x| is differentiable at x = 2

Now $f'(2) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{|2+h|-|2|}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1$

Therefore,

f'(2) = 1

10. Question

Find the values of a and b so that the function

$f(x) = \begin{cases} (x^2+3x+a), & \text{when } x \le 1; \\ (bx+2), & \text{when } x > 1 \end{cases}$ is differentiable at each x ϵ R

Answer

It is given that f(x) is differentiable at each x ϵ R

For x ≤ 1,

f(x) = x^2 + 3x + a i.e. a polynomial

for x > 1,

f(x) = bx + 2, which is also a polynomial

Since, a polynomial function is everywhere differentiable. Therefore, f(x) is differentiable for all x > 1 and for all x < 1.

f(x) is continuous at x = 1

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\operatorname{Lim}_{x\to 1}(x^2 + 3x + a) = \lim_{x\to 1}(bx + 2) = 1 + 3 + a$$

$$1^2 + 3(1) + a = b(1) + 2 = 4 + a$$

4 + a = b + 2

a – b + 2 = 0 ...(1)

As function is differentiable, therefore, LHD = RHD

LHD at x = 1:

$$\operatorname{Lim}_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{x^2+3x+a-(4+a)}{x-1} = \lim_{x\to 1}\frac{x^2+3x-4}{x-1} = \lim_{x\to 1}\frac{(x+4)(x-1)}{x-1}$$

$$= \lim_{x\to 1}(x + 4) = 1 + 4 = 5$$

RHD at x = 1:

$$\operatorname{Lim}_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{(bx+2)-(4+a)}{x-1} = \lim_{x\to 1}\frac{bx-2-a}{x-1} = \lim_{x\to 1}\frac{bx-b}{x-1} = \lim_{x\to 1}\frac{b(x-1)}{x-1}$$

$$= \lim_{x\to 1} b = b$$

As, LHD = RHD

Therefore,

5 = b

Putting b in (1), we get,

a – b + 2 = 0

a - 5 + 2 =0

a = 3

Hence,

a = 3 and b = 5