# 26. Fundamental Concepts of 3-Dimensional Geometry

## Exercise 26

### 1. Question

Find the direction cosines of a line segment whose direction ratios are:

(i) 2. – 6, 3

(ii) 2, - 1, - 2,

(iii) – 9, 6, -2

**Answer**

(i) direction ratios are:- (2, -6, 3)

So, the direction cosines are- (l, m, n), where, $l^2 + m^2 + n^2 = 1$,

So, l, m, and n are:-

$$l = \frac{2}{\sqrt{2^2 + (-6)^2 + 3^2}}$$

$$m = -\frac{6}{\sqrt{2^2 + (-6)^2 + 3^2}}$$

$$n = \frac{3}{\sqrt{2^2 + (-6)^2 + 3^2}}$$

$$(l, m, n) = (\frac{2}{7}, -\frac{6}{7}, \frac{3}{7})$$

The direction cosines are:- $(\frac{2}{7}, -\frac{6}{7}, \frac{3}{7})$

(ii) direction ratios are:- (2, -1, -2)

So, the direction cosines are:- (l, m, n), where, $l^2 + m^2 + n^2 = 1$,

So, l, m, and n are:-

$$l = \frac{2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$m = -\frac{-1}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$n = \frac{-2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$$

$$(l, m, n) = (\frac{2}{3}, -\frac{1}{3}, \frac{-2}{3})$$

The direction cosines are:- $(\frac{2}{3}, -\frac{1}{3}, \frac{-2}{3})$

(iii) direction ratios are:- (-9, 6, -2)

So, the direction cosines are- (l, m, n), where, $l^2 + m^2 + n^2 = 1$,

So, l, m, and n are:-

$$l = -\frac{9}{\sqrt{(-9)^2 + 6^2 + (-2)^2}}$$

$$m = \frac{6}{\sqrt{(-9)^2 + 6^2 + (-2)^2}}$$

$$n = \frac{-2}{\sqrt{(-9)^2 + 6^2 + (-2)^2}}$$

$$(l, m, n) = (\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11})$$

The direction cosines are:- $(\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11})$

**2. Question**

Find the direction ratios and the direction cosines of the line segment joining the points:

(i) A (1, 0, 0) and B(0, 1, 1)

(ii) A(5, 6, -3) and B (1, -6, 3)

(iii) A (-5, 7, -9) and B (-3, 4, -6)

**Answer**

Given two line segments , we have the direction ratios,

Of the line joining these 2 points as,

AB = $-\hat{\imath} + \hat{\jmath}$ + k, (direction ratio)

The unit vector in this direction will be the direction cosines, i.e.,

Unit vector in this direction is:- $(-\hat{\imath} + \hat{\jmath} + k)/\sqrt{3}$

The direction cosines are $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

(ii) Given two line segments , we have the direction ratios,

Of the line joining these 2 points as,

AB = $-4\hat{\imath} + (-12)\hat{\jmath} + 6k$

The direction ratio in the simplest form will be, (2, 6, -3)

The unit vector in this direction will be the direction cosines, i.e.,

Unit vector in this direction is:- $(2\hat{\imath} + 6\hat{\jmath} - 3k)/\sqrt{2^2 + 6^2 + (-3)^2}$

The direction cosines are $(\frac{2}{7}, \frac{6}{7}, -\frac{3}{7})$

(iii) Given two line segments , we have the direction ratios,

Of the line joining these 2 points as,

AB = $2\hat{\imath} - 3\hat{\jmath} + 3k$, (direction ratio)

The unit vector in this direction will be the direction cosines, i.e.,

Unit vector in this direction is:- $(2\hat{\imath} - 3\hat{\jmath} + 3k)/\sqrt{2^2 + (-3)^2 + 3^2}$

The direction cosines are $(\frac{2}{\sqrt{22}}, -\frac{3}{\sqrt{22}}, \frac{3}{\sqrt{22}})$

**3. Question**

Show that the line joining the points A(1,-1, 2) and B(3, 4, -2) is perpendicular to the line joining the points C(0, 3, 2) and D (3, 5, 6).

**Answer**

Given: A(1, -1, 2) and B(3, 4, -2)

The line joining these two points is given by,

AB = 2i + 5j -4k

C(0, 3, 2) and D(3, 5, 6),

The line joining these two points,

CD = 3i + 2j +4k

To prove that the two lines are perpendicular we need to show that the angle between these two lines is $\frac{\pi}{2}$

So, AB.CD = 0 (dot product)

Thus, (2i + 5j -4k). (3i + 2j +4k) = 6 + 10 – 16 = 0.

Thus, the two lines are perpendicular.

**4. Question**

Show that the line segment joining the origin to the point A(2, 1, 1) is perpendicular to the line segment joining the points B(3, 5, -1) and C(4,3, -1).

**Answer**

Given: O(0, 0, 0) and A(2, 1, 1)

The line joining these two points is given by,

OA = 2i + j +k

B(3, 5, -1) and D(4, 3, -1),

The line joining these two points,

BC = i - 2j +0k

To prove that the two lines are perpendicular we need to show that the angle between these two lines is $\frac{\pi}{2}$

So, OA.BC = 0 (dot product)

Thus, (2i + j +k). (i - 2j +0k) = 2 - 2 + 0 = 0.

Thus, the two lines are perpendicular.

**5. Question**

Find the value of p for which the line through the points A(3, 5, -1) and B(5, p, 0) 9 is perpendicular to the line through the points C(2, 1, 1) and D (3, 3, -1).

**Answer**

Given: A(3, 5, -1) and B(5, p, 0)

The line joining these two points is given by,

AB = 2i + (p-5)j + k

C(2, 1, 1) and D(3, 3, -1),

The line joining these two points,

CD = i + 2j - 2k

As the two lines are perpendicular, we know that the angle between these two lines is $\frac{\pi}{2}$

So, AB.CD = 0 (dot product)

Thus, (2i + (p-5)j + k) .(i + 2j - 2k) = 0.

ð $2 + 2(p - 5) - 2 = 0$

ð p = 5

Thus, p = 5.

**6. Question**

If O is the origin and P (2, 3,4) and Q (1, -2, 1) be any two points show that OP $\perp$ OQ.

**Answer**

Given O(0, 0, 0), P(2, 3, 4) So, OP = 2i + 3j + 4k

Q(1, -2, 1), So, OQ = i – 2j + k

To prove that OP $\perp$ OQ we have,

OP.OQ = 0, i.e. the angle between the line segments is $\frac{\Pi}{2}$

So, the dot product i.e. |OP||OQ|cos$\theta$ = 0,cos$\theta$ = 0,

OP.OQ = 0

Thus, (2i + 3j + 4k).( i – 2j + k) = 2 – 6 + 4 = 0

Hence, proved.

**7. Question**

Show that the line segment joining the points A(1, 2, 3) and B (4, 5, 7) is parallel to the segment joining the points C(-4, 3, -6) and D (2, 9, 2).

**Answer**

Given A(1, 2, 3), B(4, 5, 7), the line joining these two points will be

AB = 3i + 3j + 4k

And the line segment joining, C(-4, 3, -6) and D(2, 9, 2) will be

CD = 6i + 6j + 8k

If CD = r(AB), where r is a scalar constant then,

The two lines are parallel.

Here, CD = 2(AB),

Thus, the two lines are parallel.

**8. Question**

If the line segment joining the points A(7, p, 2) and B(q, -2, 5) be parallel to the line segment joining the points C(2, -3, 5) and D(-6, -15,11), find the values of p and q.

**Answer**

Given: A(7, p, 2) and B(q, -2, 5), line segment joining these two points will be, AB = (q-7)i + (-2-p)j + 3k

And the line segment joining C(2, -3, 5) and D(-6, -15, 11) will be, CD = -8i – 12j + 6k

Then, the angle between these two line segments will be 0 degree. So, the cross product will be 0.

AB$\times$CD = 0

ð ((q-7)i + (-2-p)j + 3k)×( -8i – 12j + 6k) = 0

Thus, solving this we get,

p = 4 and q = 3

**9. Question**

Show that the points A(2, 3, 4), B(-1, -2, 1) and C (5, 8, 7) are collinear.

**Answer**

We have to show that the three points are colinear , i.e. they all lie on the same line,

If we define a line which is having a parallel line to AB and the points A and B lie on it, if point C also satisfies the line then, the three points are colinear,

Given A(2, 3, 4) and B(-1, -2, 1), AB = -3i – 5j -3k

The points on the line AB with A on the line can be written as,

R = (2, 3, 4) +a(-3, -5, -3)

Let C = (2-3a, 3-5a, 4-3a)

ð (5, 8, 7) = (2-3a, 3-5a, 4-3a)

ð If a = -1, then L.H.S = R.H.S, thus

The point C lies on the line joining AB,

Hence, the three points are colinear.

**10. Question**

Show that the points A(-2, 4.7), B(3, -6. -8) and C(1, -2, -2) are collinear.

**Answer**

We have to show that the three points are colinear , i.e. they all lie on the same line,

If we define a line which is having a parallel line to AB and the points A and B lie on it, if point C also satisfies the line then, the three points are colinear,

Given A(-2, 4, 7) and B(3, -6, -8), AB = 5i – 10j -15k

The points on the line AB with A on the line can be written as,

R = (-2, 4, 7) +a(5, -10, -15)

Let C = (-2+5a, 4-10a, 7-15a)

ð (1, -2, -2) = (-2+5a, 4-10a, 7-15a)

ð If a = 3/5, then L.H.S = R.H.S, thus

The point C lies on the line joining AB,

Hence, the three points are colinear.

**11. Question**

Find the value of p for which the points A(-1, 3, 2), B(-4, 2, -2), and C(5, 5, p) are collinear.

**Answer**

We have to show that the three points are colinear , i.e. they all lie on the same line,

If we define a line which is having a parallel line to AB and the points A and B lie on it, as the points are colinear so C must satisfy the line,

Given A(-1, 3, 2) and B(-4, 2, -2), AB = -3i – j -4k

The points on the line AB with A on the line can be written as, R = (-1, 3, 2) +a(-3, -1, -4)

Let C = (-1-3a, 3-1a, 2-4a)

ð (5, 5, p) = (-1-3a, 3-1a, 2-4a)

ð As L.H.S = R.H.S, thus

ð 5 = -1 – 3a, a = -2

Substituting a = -2 we get, p = 2-4(-2) = 10

Hence, p = 10.

**12. Question**

Find the angle between the two lines whose direction cosines are:

$\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$ and $\frac{3}{7}, \frac{2}{7}, \frac{6}{7}$

**Answer**

Let

$R_1 = \frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k$

And $R_2 = \frac{3}{7}i + \frac{2}{7}j + \frac{6}{7}k$

$R_1.R_2 = |R_1||R_2|\cos\theta$

Here, as R1 and R2 are the unit vectors with a direction given by the direction cosines hence, |R1| and |R2| are 1.

So, $\cos\theta = R_1.R_2 / 1$

ð $\cos\theta = \frac{6}{21} - \frac{2}{21} - \frac{12}{21} = \frac{8}{21}$

ð $\theta = \cos^{-1} - \frac{8}{21}$

The angle between the lines is $\cos^{-1} - \frac{8}{21}$

**13. Question**

Find the angle between the two lines whose direction ratios are:

a, b, c and (b – c), (c – a), (a – b).

**Answer**

The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where $R_1$ an $R_2$ denote the vectors with the direction ratios,

So, here we have,

$R_1$ = ai + bj + ck and $R_2$ = (b-c)i +(c-a)j + (a-b)k

$\cos\theta = \frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}} = 0$

$\cos\theta = 0$

Hence, $\theta = \frac{\pi}{2}$

**14. Question**

Find the angle between the lines whose direction ratios are:

2, -3, 4 and 1, 2, 1.

**Answer**

The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where $R_1$ and $R_2$ denote the vectors with the direction ratios,

So, here we have,

$R_1$ = 2i - 3j + 4k and $R_2$ = i +2j + k

$\cos\theta = \frac{2-6+4}{\sqrt{2^2+(-3)^2+4^2}\sqrt{1^2+2^2+1^2}} = 0$

$\cos\theta = 0$

Hence, $\theta = \frac{\pi}{2}$

**15. Question**

Find the angle between the lines whose direction ratios are:

1, 1, 2 and $(\sqrt{3-1}),(-\sqrt{3-1}),4$

**Answer**

The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where $R_1$ and $R_2$ denote the vectors with the direction ratios,

So, here we have,

R1 = i + j + 2k and R2 = $(\sqrt{3}-1)$i -$(\sqrt{3}+1)$j + (4)k

$\cos\theta = \frac{\sqrt{3}-1-\sqrt{3}-1+8}{\sqrt{1^2+1^2+2^2}\sqrt{(\sqrt{3}-1)^2+(-(\sqrt{3}+1))^2+4^2}} = \frac{6}{\sqrt{6}.\sqrt{24}}$

$\cos\theta = \frac{1}{2}$

Hence, $\theta = \frac{\pi}{3}$

**16. Question**

Find the angle between the vectors $\overrightarrow{r_1} = (3\hat{i} - 2\hat{j} + \hat{k})$ and $\overrightarrow{r_2} = (4\hat{i} + 5\hat{j} + 7\hat{k})$

**Answer**

The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where $R_1$ and $R_2$ denote the vectors with the direction ratios,

So, here we have,

R1 = 3i - 2j + k and R2 = 4i + 5j + 7k

$\cos\theta = \frac{12-10+7}{\sqrt{3^2+(-2)^2+1^2}\sqrt{4^2+5^2+7^2}} = \frac{9}{\sqrt{14}.\sqrt{90}}$

$\cos\theta = \frac{3}{2\sqrt{35}}$

Hence, $\theta = \cos^{-1}\frac{3}{2\sqrt{35}}$

**17. Question**

Find the angles made by the following vectors with the coordinate axes:

(i) $(\hat{i}-\hat{j}+\hat{k})$

(ii) $(\hat{j}-\hat{k})$

(iii) $(\hat{i}-4\hat{j}+8\hat{k})$

**Answer**

(i) The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where $R_1$ and $R_2$ denote the vectors with the direction ratios,

So, here we have,

R1 = i - j + k and R2 = i for x- axis

$\cos\theta = \frac{1-0+0}{\sqrt{1^2+(-1)^2+1^2}\sqrt{1^2}} = \frac{1}{\sqrt{3}}$

$\cos\theta = \frac{1}{\sqrt{3}}$

Hence, $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$

With y- axis, i. e. R2 = j

$\cos\theta = \frac{0-1+0}{\sqrt{1^2+(-1)^2+1^2}\sqrt{1^2}} = -\frac{1}{\sqrt{3}}$

$\cos\theta = -\frac{1}{\sqrt{3}}$

Hence, $\theta = \cos^{-1}(-\frac{1}{\sqrt{3}})$

With z- axis, i. e. R2 = k

$\cos\theta = \frac{0-0+1}{\sqrt{1^2+(-1)^2+1^2}\sqrt{1^2}} = \frac{1}{\sqrt{3}}$

$\cos\theta = \frac{1}{\sqrt{3}}$

Hence, $\theta = \cos^{-1}(\frac{1}{\sqrt{3}})$

(ii) The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where $R_1$ and $R_2$ denote the vectors with the direction ratios,

So, here we have,

R1 = j - k and R2 = i for x- axis

$\cos\theta = \frac{0-0+0}{\sqrt{0^2+1^2+(-1)^2}\sqrt{1^2}} = 0$

$\cos\theta = 0$

Hence, $\theta = \frac{\pi}{2}$

With y- axis, i. e. R2 = j

$\cos\theta = \frac{0+1+0}{\sqrt{0^2+1^2+(-1)^2}\sqrt{1^2}} = \frac{1}{\sqrt{2}}$

$\cos\theta = \frac{1}{\sqrt{2}}$

Hence, $\theta = \frac{\pi}{4}$

With z- axis, i. e. R2 = k

$\cos\theta = \frac{0+0-1}{\sqrt{0^2+1^2+-(1)^2}\sqrt{1^2}} = -\frac{1}{\sqrt{2}}$

$\cos\theta = -\frac{1}{\sqrt{2}}$

Hence, $\theta = \frac{3\pi}{4}$

(iii) The angle between the two lines is given by

$\cos\theta = \frac{R_1.R_2}{|R_1||R_2|}$

where $R_1$ and $R_2$ denote the vectors with the direction ratios,

So, here we have,

R1 = i - 4j + 8k and R2 = i for x- axis

$\cos\theta = \frac{1-0+0}{\sqrt{1^2+(-4)^2+8^2}\sqrt{1^2}} = \frac{1}{\sqrt{81}}$

$\cos\theta = \frac{1}{9}$

Hence, $\theta = \cos^{-1}\frac{1}{9}$

With y- axis, i. e. R2 = j

$\cos\theta = \frac{0-4+0}{\sqrt{1^2+(-4)^2+8^2}\sqrt{1^2}} = -\frac{4}{9}$

$\cos\theta = -\frac{1}{9}$

Hence, $\theta = \cos^{-1}(-\frac{1}{9})$

With z- axis, i. e. R2 = k

$\cos\theta = \frac{0-0+8}{\sqrt{1^2+(-4)^2+8^2}\sqrt{1^2}} = \frac{8}{9}$

$\cos\theta = \frac{8}{9}$

Hence, $\theta = \cos^{-1}(\frac{8}{9})$

**18. Question**

Find the coordinates of the foot of the perpendicular drawn from the point A(1, 8, 4) to the line joining the points B(0, -1, 3) and C(2, -3, -1).

**Answer**

Given: A(1, 8, 4)

Line segment joining B(0, -1, 3) and C(2, -3, -1) is

BC = 2i – 2j – 4k

Let the foot of the perpendicular be R then,

As R lies on the line having point B and parallel to BC,

So, R = (0, -1, 3) + a(2, -2, -4)

R(2a, -1-2a, 3-4a)

The line segment AR is

AR = (2a-1)i + (-1-2a-8)j + (3-4x-4)k

As the lines AR and BC are perpendicular thus, (as R is the foot of the perpendicular on BC)

AR.BC = 0

ð 2(2a-1) + (-2)(-9-2a) + (-4)(-1-4a) = 0

ð 24a + 20 = 0

ð a = $-\frac{5}{6}$

Substituting a in R we get,

R$(-\frac{5}{3}, \frac{2}{3}, \frac{19}{3})$