Principle Of Mathematical Induction

Exercise 4

Q. 1. Using the principle of mathematical induction, prove each of the following for all n ϵ N:
1 + 2 + 3 + 4 + … + n = 1/2 n(n + 1)

Answer : To Prove:

1 + 2 + 3 + 4 + … + n = 1/2 n(n + 1)

Steps to prove by mathematical induction:

Let P(n) be a statement involving the natural number n such that

(i) P(1) is true

(ii) P(k + 1) is true, whenever P(k) is true

Then P(n) is true for all n ϵ N

Therefore,

Let P(n): 1 + 2 + 3 + 4 + … + n = 1/2 n(n + 1)

Step 1:

P(1) = 1/2 1(1 + 1) = 1/2 × 2 = 1

Therefore, P(1) is true

Step 2:

Let P(k) is true Then,

P(k): 1 + 2 + 3 + 4 + … + k = 1/2 k(k + 1)

Now,

1 + 2 + 3 + 4 + … + k + (k + 1) = 1/2 k(k + 1) + (k + 1)

= (k + 1){ 1/2 k + 1}

= 1/2 (k + 1) (k + 2)

= P(k + 1)

Hence, P(k + 1) is true whenever P(k) is true

Hence, by the principle of mathematical induction, we have

1 + 2 + 3 + 4 + … + n = 1/2 n(n + 1) for all n ϵ N

Hence proved.

Q. 2. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

2 + 4 + 6 + 8 + …. + 2n = n(n + 1)

Answer : To Prove:

2 + 4 + 6 + 8 + …. + 2n = n(n + 1)

Steps to prove by mathematical induction:

Let P(n) be a statement involving the natural number n such that

(i) P(1) is true

(ii) P(k + 1) is true, whenever P(k) is true

Then P(n) is true for all n ϵ N

Therefore,

Let P(n): 2 + 4 + 6 + 8 + …. + 2n = n(n + 1)

Step 1:

P(1) = 1(1 + 1) = 1 × 2 = 2

Therefore, P(1) is true

Step 2:

Let P(k) is true Then,

P(k): 2 + 4 + 6 + 8 + …. + 2k = k(k + 1)

Now,

2 + 4 + 6 + 8 + …. + 2k + 2(k + 1) = k(k + 1) + 2(k + 1)

= k(k + 1) + 2(k + 1)

= (k + 1) (k + 2)

= P(k + 1)

Hence, P(k + 1) is true whenever P(k) is true

Hence, by the principle of mathematical induction, we have

2 + 4 + 6 + 8 + … + 2n = n(n + 1) for all n ϵ N

Q. 3. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$$1 + 3 + 3^2 + 3^3 + \ldots + 3^{n-1} = \frac{1}{2}(3^n - 1)$$

Answer : To Prove:

$$1 + 3^1 + 3^2 + \ldots + 3^{n-1} = \frac{3^n - 1}{2}$$

Steps to prove by mathematical induction:

Let P(n) be a statement involving the natural number n such that

(i) P(1) is true

(ii) P(k + 1) is true, whenever P(k) is true

Then P(n) is true for all n ϵ N

Therefore,

Let P(n): $1 + 3^1 + 3^2 + \ldots + 3^{n-1} = \frac{3^n - 1}{2}$

Step 1:

P(1) = $\frac{3^1 - 1}{2} = \frac{2}{2}$ = 1

Therefore, P(1) is true

Step 2:

Let P(k) is true Then,

P(k): $1 + 3^1 + 3^2 + \ldots + 3^{k-1} = \frac{3^k - 1}{2}$

Now,

$$1 + 3^1 + 3^2 + \ldots + 3^{k-1} + 3^{(k+1)-1} = \frac{3^{(k)} - 1}{2} + 3^{(k+1)-1}$$

$= \frac{3^k - 1}{2} + 3^{(k)}$

$= 3^{(k)}(\frac{1}{2} + 1) - \frac{1}{2}$

$= 3^{(k)}(\frac{3}{2}) - \frac{1}{2}$

$= 3^{(k+1)}(\frac{1}{2}) - \frac{1}{2}$

$= \frac{3^{(k+1)} - 1}{2}$

= P(k + 1)

Hence, P(k + 1) is true whenever P(k) is true

Hence, by the principle of mathematical induction, we have

$1 + 3^1 + 3^2 + \ldots + 3^{n-1} = \frac{3^n - 1}{2}$ for all n ϵ N

Q. 4. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$2 + 6 + 18 + \ldots + 2 \times 3^{n-1} = (3^n - 1)$

Answer : To Prove:

$2 + 6 + 18 + \ldots + 2 \times 3^{n-1} = (3^n - 1)$

Steps to prove by mathematical induction:

Let P(n) be a statement involving the natural number n such that

(i) P(1) is true

(ii) P(k + 1) is true, whenever P(k) is true

Then P(n) is true for all $n \in N$

Therefore,

Let P(n): $2 + 6 + 18 + \ldots + 2 \times 3^{n-1} = (3^n - 1)$

Step 1:

$P(1) = 3^1 - 1 = 3 - 1 = 2$

Therefore, P(1) is true

Step 2:

Let P(k) is true Then,

P(k): $2 + 6 + 18 + \ldots + 2 \times 3^{k-1} = (3^k - 1)$

Now,

$2 + 6 + 18 + \ldots + 2 \times 3^{k-1} + 2 \times 3^{k+1-1} = (3^k - 1) + 2 \times 3^k$

$= - 1 + 3 \times 3^k$

$= 3^{k+1-} 1$

$= P(k + 1)$

Hence, P(k + 1) is true whenever P(k) is true

Hence, by the principle of mathematical induction, we have

$2 + 6 + 18 + \ldots + 2 \times 3^{n-1} = (3^n - 1)$ for all $n \in N$

Q. 5. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^n}=\left(1-\frac{1}{2^n}\right)$$

Answer : To Prove:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^n}=\left(1-\frac{1}{2^n}\right)$$

Steps to prove by mathematical induction:

Let P(n) be a statement involving the natural number n such that

(i) P(1) is true

(ii) P(k + 1) is true, whenever P(k) is true

Then P(n) is true for all n ϵ N

Therefore,

Let P(n): $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^n}=\left(1-\frac{1}{2^n}\right)$

Step 1:

P(1) = $1-\frac{1}{2^1} = 1-\frac{1}{2} = \frac{1}{2}$

Therefore, P(1) is true

Step 2:

Let P(k) is true Then,

P(k):$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^k} = 1-\frac{1}{2^k}$

Now,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$

$= 1 + \frac{1}{2^k}\left(\frac{1}{2} - 1\right)$

$= 1 + \frac{1}{2^k}(-\frac{1}{2})$

$= 1 - \frac{1}{2^{k+1}}$

= P(k + 1)

Hence, P(k + 1) is true whenever P(k) is true

Hence, by the principle of mathematical induction, we have

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = \left(1 - \frac{1}{2^n}\right)$ for all $n \in N$

Q. 6. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$1^2 + 3^2 + 5^2 + 7^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Answer : To Prove:

$1^2 + 3^2 + 5^2 + 7^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Steps to prove by mathematical induction:

Let P(n) be a statement involving the natural number n such that

(i) P(1) is true

(ii) P(k + 1) is true, whenever P(k) is true

Then P(n) is true for all n ϵ N

Therefore,

Let P(n): $1^2 + 3^2 + 5^2 + 7^2 + \ldots + (2n - 1)^2 = \frac{n(2n-1)(2n+1)}{3}$

Step 1:

P(1) = $\frac{1(2-1)(2+1)}{3} = \frac{3}{3} = 1$

Therefore, P(1) is true

Step 2:

Let P(k) is true Then,

P(k): $1^2 + 3^2 + 5^2 + 7^2 + \ldots + (2k - 1)^2 = \frac{k(2k-1)(2k+1)}{3}$

Now,

$1^2 + 3^2 + 5^2 + 7^2 + \ldots + (2(k + 1)-1)^2 = \frac{k(2k-1)(2k+1)}{3} + (2k + 2 - 1)^2$

$= \frac{k(2k-1)(2k+1)}{3} + (2k + 1)^2$

$= (2k + 1)[\frac{k(2k-1)}{3} + 2k + 1]$

$= (2k + 1)[\frac{2k^2 - k + 6k + 3}{3}]$

$= (2k + 1)[\frac{2k^2 + 5k + 3}{3}]$

$= (2k + 1)[\frac{(k+1)(2k+3)}{3}]$ (Splitting the middle term)

$= \frac{(k+1)(2k+1)(2k+3)}{3}$

= P(k + 1)

Hence, P(k + 1) is true whenever P(k) is true

Hence, by the principle of mathematical induction, we have

$1^2 + 3^2 + 5^2 + 7^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$ for all n ϵ N

Q. 7. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$1.2 + 2.2^2 + 3.2^3 + \ldots . + n.2^n = (n-1)2^{n+1} + 2$.

Answer : To Prove:

$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \ldots\ldots + n \times 2^n = (n-1)2n + 1 + 2$

Let us prove this question by principle of mathematical induction (PMI)

Let P(n): $1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \ldots\ldots + n \times 2^n$

For n = 1

LHS = 1 × 2 = 2

RHS = $(1 - 1) \times 2^{(1+1)} + 2$

= 0 + 2 = 2

Hence, LHS = RHS

P(n) is true for n 1

Assume P(k) is true

$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + k \times 2^k = (k-1) \times 2^{k+1} + 2$ ……(1)

We will prove that P(k + 1) is true

$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + (k+1) \times 2^{k+1} = ((k+1) - 1) \times 2^{(k+1)+1} + 2$

$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + (k+1) \times 2^{k+1} = (k) \times 2^{k+2} + 2$

1×$2^1 + 2 \times 2^2 + 3 \times 2^3 + k2^k + (k + 1) \times 2^{k+1} = (k) \times 2^{k+2} + 2$ ……(2)

We have to prove P(k + 1) from P(k), i.e. (2) from (1)

From (1)

1×$2^1 + 2 \times 2^2 + 3 \times 2^3 + k \times 2^k = (k - 1) \times 2^{k+1} + 2$

Adding $(k + 1) \times 2^{k+1}$ both sides,

(1×
$2^1 + 2 \times 2^2 + 3 \times 2^3 + k \times 2^k) + (k + 1) \times 2^{k+1} = (k - 1) \times 2^{k+1} + 2$ +
$(k + 1) \times 2^{k+1}$

= $k \times 2^{k+1} - 2^{k+1} + 2 + k \times 2^{k+1} + 2^{k+1}$

= $2k \times 2^{k+1} + 2$

= $k \times 2^{k+2} + 2$

(1×$2^1 + 2 \times 2^2 + 3 \times 2^3 + k \times 2^k) + (k + 1) \times 2^{k+1}$ = $k \times 2^{k+2} + 2$

Which is the same as P(k + 1)

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for

Where n is a natural number

Put k = n - 1

(1× $2^1 + 2 \times 2^2 + 3 \times 2^3) + n \times 2^n$ = $(n - 1) \times 2^{n+1} + 2$

Hence proved.

Q. 8. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$3.2^2 + 3^2.2^3 + 3^3.2^4 + \ldots + 3^n.2^{n+1} = \frac{12}{5}(6^n - 1)$.

Answer : To Prove:

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \ldots\ldots \times + 3^n \times 2^{n+1} = \frac{12}{5}(6^n - 1)$$

Let us prove this question by principle of mathematical induction (PMI)

Let P(n): $3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \ldots\ldots \times + 3^n \times 2^{n+1}$

For n = 1

LHS = 3×2^2 = 12

RHS = $\left(\frac{12}{5}\right) \times (6^1 - 1)$

= $\frac{12}{5} \times 5$ = 12

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \ldots\ldots \times + 3^k \times 2^{k+1} = \frac{12}{5}(6^k - 1) \quad \ldots\ldots(1)$$

We will prove that P(k + 1) is true

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \ldots\ldots \times + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^{k+1} - 1)$$

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \ldots\ldots \times + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^{k+1}) - \frac{12}{5}$$

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \ldots\ldots \times + 3^k \times 2^{k+1} + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^{k+1}) - \frac{12}{5}$$

…(2)

We have to prove P(k + 1) from P(k) ie (2) from (1)

From (1)

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots\dots \times + 3^k \times 2^{k+1} = \frac{12}{5}(6^k - 1)$$

Adding $3^{k+1} \times 2^{k+2}$ both sides

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots\dots + 3^k \times 2^{k+1} + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^k - 1) + 3^{k+1} \times 2^{k+2}$$

$$= \frac{12}{5}(6^k - 1) + 3^k \times 2^k \times 12$$

$$= \frac{12}{5}(6^k - 1) + 6^k \times 12$$

$$= \left(6^k(\frac{12}{5} + 12) - \frac{12}{5}\right)$$

$$= \left(\frac{72}{5}\right) - \frac{12}{5}$$

$$= \frac{12}{5}(6^{k+1}) - \frac{12}{5}$$

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots\dots + 3^k \times 2^{k+1} + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^{k+1}) - \frac{12}{5}$$

Which is the same as P(k + 1)

Therefore, P (k + 1) is true whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for×

Where n is a natural number

Put k = n - 1

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots\dots \times + 3^n \times 2^{n+1} = \frac{12}{5}(6^n) - \frac{12}{5}$$

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots\dots \times + 3^n \times 2^{n+1} = \frac{12}{5}(6^n - 1)$$

Hence proved

Q. 9. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$$1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \dots + $$

Answer : To Prove:

$$\frac{1}{1} + \frac{1}{(1 + 2)} + \frac{1}{(1 + 2 + 3 + \dots\dots + n)} = \frac{2n}{(n + 1)}$$

Let us prove this question by principle of mathematical induction (PMI)

Let P(n): $\frac{1}{1} + \frac{1}{(1+2)} + \frac{1}{(1+2+3+\dots\dots+n)} = \frac{2n}{(n+1)}$

For n = 1

LHS = 1

RHS = $\frac{2\times 1}{(1+1)} = 1$

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

$$\frac{1}{1} + \frac{1}{(1+2)} + \dots\dots + \frac{1}{(1+2+3+\dots\dots\times + k)} = \frac{2k}{(k+1)} \quad \dots\dots(1)$$

We will prove that P(k + 1) is true

RHS = $\frac{2(k+1)}{(k+1+1)} = \frac{2k+2}{k+2}$

LHS = $\frac{1}{1} + \frac{1}{(1+2)} + \dots\dots + \frac{1}{(1+2+3+\dots\dots+(k+1))}$

$= \frac{1}{1} + \frac{1}{(1+2)} + \dots\dots + \frac{1}{(1+2+3+\dots\dots+k)} + \frac{1}{(1+2+3+\dots\dots+(k+1))}$ [Writing the last

Second term]

$= \frac{2k}{(k+1)} + \frac{1}{(1+2+3+\dots\dots+(k+1))}$ [From 1]

$$= \frac{2k}{(k+1)} + \frac{1}{\frac{(k+1)\times(k+2)}{2}}$$

{ 1 + 2 + 3 + 4 + … + n = [n(n + 1)]/2 put n = k + 1 }

$$= \frac{2k}{(k+1)} + \frac{2}{(k+1)\times(k+2)}$$

$$= \frac{2}{(k+1)}\left(\frac{k}{1} + \frac{1}{k+2}\right)$$

$= \frac{2}{k+1}\left(\frac{(k+1)\times(k+1)}{k+2}\right)$

[Taking LCM and simplifying]

$= \frac{2(k+1)}{(k+2)}$

= RHS

Therefore, $\frac{1}{1} + \frac{1}{(1+2)} + \dots\dots + \frac{1}{(1+2+3+\dots\dots\times+(k+1))} = \frac{2k+2}{k+2}$

LHS = RHS

Therefore, P (k + 1) is true whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for×

Where n is a natural number

Put k = n - 1

$$\frac{1}{1} + \frac{1}{(1 + 2)} + \dots\dots + \frac{1}{(1 + 2 + 3 + \dots\dots \times + n)} = \frac{2n}{n + 1}$$

Hence proved

Q. 10. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots\dots + \frac{1}{(3n - 1) \times (3n + 2)} = \frac{n}{(6n + 4)}$$

Answer : To Prove:

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots\dots + \frac{1}{(3n - 1) \times (3n + 2)} = \frac{n}{(6n + 4)}$$

For n = 1

LHS = $\frac{1}{2\times5} = \frac{1}{10}$

RHS = $\frac{1\times1}{(6+4)} = \frac{1}{10}$

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

$$\frac{1}{2\times5} + \frac{1}{(5\times8)} + \dots\dots + \frac{1}{(3k-1)\times(3k+2)} = \frac{k}{(6k+4)} \quad \dots\dots(1)$$

We will prove that P(k + 1) is true

RHS = $\frac{k+1}{(6(k+1)+4)} = \frac{k+1}{(6k+10)}$

LHS = $\frac{1}{2\times5} + \frac{1}{(5\times8)} + \dots\dots + \frac{1}{(3k-1)\times(3k+2)} + \frac{1}{(3(k+1)-1)\times(3(k+1)+2)}$

[Writing the Last second term]

$= \frac{1}{2\times5} + \frac{1}{(5\times8)} + \dots\dots + \frac{1}{(3k-1)\times(3k+2)} + \frac{1}{(3(k+1)-1)\times(3(k+1)+2)}$

$= \frac{k}{(6k+4)} + \frac{1}{(3(k+1)-1)\times(3(k+1)+2)}$ [Using 1]

$= \frac{k}{(6k+4)} + \frac{1}{(3k+2)\times(3k+5)}$

$= \frac{k}{(6k+4)} + \frac{1}{(3k+2)\times(3k+5)}$

$= \frac{1}{(3k+2)} \times [\frac{(3k+2)\times(k+1)}{2\times(3k+5)}]$ (Taking LCM and simplifying)

$= \frac{k+1}{(6k+10)}$

= RHS

Therefore, $\frac{1}{2\times5} + \frac{1}{(5\times8)} + \dots\dots + \frac{1}{(3k-1)\times(3k+2)} + \frac{1}{(3(k+1)-1)\times(3(k+1)+2)} = \frac{k+1}{(6k+10)}$

LHS = RHS

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for

Where n is a natural number

Put k = n - 1

$\frac{1}{2\times5} + \frac{1}{(5\times8)} + \dots\dots \times + \frac{1}{(3n-1)\times(3n+2)} = \frac{n}{(6n+4)}$

Hence proved.

Q. 11. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}.$$

Answer : To Prove:

$$\frac{1}{1 \times 4} + \frac{1}{(4 \times 7)} + \dots \dots + \frac{1}{(3n-2) \times (3n + 1)} = \frac{n}{(3n + 1)}$$

Let us prove this question by principle of mathematical induction (PMI)

Let P(n): $\frac{1}{1\times4} + \frac{1}{(4\times7)} + \dots\dots + \frac{1}{(3n-2)\times(3n+1)} = \frac{n}{(3n+1)}$

For n = 1

LHS = $\frac{1}{1\times4} = \frac{1}{4}$

RHS = $\frac{1}{(3+1)} = \frac{1}{4}$

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

= $\frac{1}{1\times4} + \frac{1}{(4\times7)} + \dots\dots + \frac{1}{(3k-2)\times(3k+1)} = \frac{k}{(3k+1)}$ ……(1)

We will prove that P(k + 1) is true

RHS = $\frac{k+1}{(3(k+1)+1)} = \frac{k+1}{(3k+4)}$

LHS = $\frac{1}{1\times4} + \frac{1}{(4\times7)} + \dots\dots + \frac{1}{(3(k+1)-2)\times(3(k+1)+1)}$

= $\frac{1}{1\times4} + \frac{1}{(4\times7)} + \dots\dots + \frac{1}{(3k-2)\times(3k+1)} + \frac{1}{(3k+1)\times(3k+4)}$

[Writing the second last term]

= $\frac{k}{(3k+1)} + \frac{1}{(3k+1)\times(3k+4)}$ [Using 1]

= $\frac{1}{(3k+1)}(k + \frac{1}{(3k+4)})$

$= \frac{1}{(3k+1)}\left(\frac{(3k^2+4k+1)}{(3k+4)}\right)$

$= \frac{k+1}{(3k+4)}$

(Splitting the numerator and cancelling the common factor)

= RHS

LHS = RHS

Therefore, P (k + 1) is true whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for

Where n is a natural number

Hence proved.

Q. 12. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{(2n+1)}.$$

Answer : To Prove:

$$\frac{1}{1\times 3}+\frac{1}{(3\times 5)}+\ldots\ldots+\frac{1}{(2n-1)\times(2n+1)}=\frac{n}{(2n+1)}$$

Let us prove this question by principle of mathematical induction (PMI)

Let P(n): $\frac{1}{1\times3}+\frac{1}{(3\times5)}+\ldots\ldots+\frac{1}{(2n-1)\times(2n+1)}=\frac{n}{(2n+1)}$

For n = 1

LHS = $\frac{1}{1\times3}=\frac{1}{3}$

RHS = $\frac{1}{(2+1)}=\frac{1}{3}$

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

$= \frac{1}{1\times3} + \frac{1}{(3\times5)} + \ldots\ldots + \frac{1}{(2k-1)\times(2k+1)} = \frac{k}{(2k+1)}$ ……(1)

We will prove that P(k + 1) is true

RHS = $\frac{k+1}{(2(k+1)+1)} = \frac{k+1}{(2k+3)}$

LHS = $\frac{1}{1\times3} + \frac{1}{(3\times5)} + \ldots\ldots + \frac{1}{(2(k+1)-1)\times(2(k+1)+1)}$

$= \frac{1}{1\times3} + \frac{1}{(3\times5)} + \ldots\ldots + \frac{1}{(2k-1)\times(2k+1)} + \frac{1}{(2k+1)\times(2k+3)}$

[Writing the second last term]

$= \frac{k}{(2k+1)} + \frac{1}{(2k+1)\times(2k+3)}$ [Using 1]

$= \frac{1}{(2k+1)}(k + \frac{1}{(2k+3)})$

$= \frac{1}{(2k+1)}(\frac{(2k^2+3k+1)}{(2k+3)})$

$= \frac{k+1}{(2k+3)}$

(Splitting the numerator and cancelling the common factor)

= RHS

LHS = RHS

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for×

Where n is a natural number

Hence proved.

Q. 13. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots\dots + \frac{1}{(3n-1) \times (3n + 2)} = \frac{n}{(6n + 4)}$$

Answer : To Prove:

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots\dots + \frac{1}{(3n-1) \times (3n + 2)} = \frac{n}{(6n + 4)}$$

For n = 1

LHS = $\frac{1}{2\times5} = \frac{1}{10}$

RHS = $\frac{1\times1}{(6+4)} = \frac{1}{10}$

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

$\frac{1}{2\times5} + \frac{1}{(5\times8)} + \dots\dots + \frac{1}{(3k-1)\times(3k+2)} = \frac{k}{(6k+4)}$ ……(1)

We will prove that P(k + 1) is true

RHS = $\frac{k+1}{(6(k+1)+4)} = \frac{k+1}{(6k+10)}$

LHS = $\frac{1}{2\times5} + \frac{1}{(5\times8)} + \dots\dots + \frac{1}{(3k-1)\times(3k+2)} + \frac{1}{(3(k+1)-1)\times(3(k+1)+2)}$[Writing the Last second term]

= $\frac{1}{2\times5} + \frac{1}{(5\times8)} + \dots\dots + \frac{1}{(3k-1)\times(3k+2)} + \frac{1}{(3(k+1)-1)\times(3(k+1)+2)}$

= $\frac{k}{(6k+4)} + \frac{1}{(3(k+1)-1)\times(3(k+1)+2)}$ [Using 1]

$= \frac{k}{(6k+4)} + \frac{1}{(3k+2)\times(3k+5)}$

$= \frac{k}{(6k+4)} + \frac{1}{(3k+2)\times(3k+5)}$

$= \frac{1}{(3k+2)} \times [\frac{(3k+2)\times(k+1)}{2\times(3k+5)}]$ (Taking LCM and simplifying)

$= \frac{k+1}{(6k+10)}$

= RHS

Therefore, $\frac{1}{2\times5} + \frac{1}{(5\times8)} + \ldots\ldots + \frac{1}{(3k-1)\times(3k+2)} + \frac{1}{(3(k+1)-1)\times(3(k+1)+2)} = \frac{k+1}{(6k+10)}$

LHS = RHS

Therefore, P (k + 1) is true whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for

Where n is a natural number

Put k = n - 1

$\frac{1}{2\times5} + \frac{1}{(5\times8)} + \ldots\ldots\times + \frac{1}{(3n-1)\times(3n+2)} = \frac{n}{(6n+4)}$

Hence proved.

Q. 14. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left\{1+\frac{(2n+1)}{n^2}\right\}$ = $(n + 1)^2$.

Answer : To Prove:

$$\left(1+\frac{3}{1}\right)\times\left(1+\frac{5}{4}\right)\times\left(1+\frac{7}{9}\right)\times \dots\dots \times\left\{1+\frac{2n+1}{n^2}\right\} = (n+1)^2$$

Let us prove this question by principle of mathematical induction (PMI)

Let P(n): $\left(1+\frac{3}{1}\right)\times\left(1+\frac{5}{4}\right)\times\left(1+\frac{7}{9}\right)\times \dots\dots \times\left\{1+\frac{2n+1}{n^2}\right\} = (n+1)^2$

For n = 1

LHS = $1+\frac{3}{1} = 4$

RHS = $(1+1)^2$ = 4

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

= $\left(1+\frac{3}{1}\right)\times\left(1+\frac{5}{4}\right)\times\left(1+\frac{7}{9}\right)\times \dots\dots \times\left\{1+\frac{2k+1}{k^2}\right\} = (k+1)^2$ …(1)

We will prove that P(k + 1) is true

RHS = $((k+1)+1)^2 = (k+2)^2$

LHS = $\left(1+\frac{3}{1}\right)\times\left(1+\frac{5}{4}\right)\times\left(1+\frac{7}{9}\right)\times \dots\dots \times\left\{1+\frac{2(k+1)+1}{(k+1)^2}\right\}$

[Now writing the second last term]

= $\left(1+\frac{3}{1}\right)\times\left(1+\frac{5}{4}\right)\times\left(1+\frac{7}{9}\right)\times \dots\dots \times\left\{1+\frac{2k+1}{k^2}\right\}\times\left\{1+\frac{2(k+1)+1}{(k+1)^2}\right\}$

= $(k+1)^2\times\left\{1+\frac{2(k+1)+1}{(k+1)^2}\right\}$ [Using 1]

= $(k+1)^2\times\left\{1+\frac{(2k+3)}{(k+1)^2}\right\}$

= $(k+1)^2\times\left\{\frac{(k+1)^2+(2k+3)}{(k+1)^2}\right\}$

$= (k + 1)^2 + (2k + 3)$

$= k^2 + 2k + 1 + 2k + 3$

$= (k + 2)^2$

= RHS

LHS = RHS

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for

Where n is a natural number

Hence proved.

Q. 15. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)....\left\{1+\frac{1}{n}\right\}$ **= (n + 1).**

Answer : To Prove:

$$\left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \ \dots\dots \times \{1 + \frac{1}{n^1}\} = (n + 1)^1$$

Let us prove this question by principle of mathematical induction (PMI)

Let P(n): $\left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \ \dots\dots \times \{1 + \frac{1}{n^1}\} = (n + 1)^1$

For n = 1

LHS = $1 + \frac{1}{1} = 2$

RHS = $(1 + 1)^1$ = 2

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

$= \left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \ldots \ldots \times \{1 + \frac{1}{k^1}\} = (k + 1)^1$ …(1)

We will prove that P(k + 1) is true

RHS = $((k + 1) + 1)^1 = (k + 2)^1$

LHS = $\left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \ldots \ldots \times \{1 + \frac{1}{(k+1)^1}\}$

[Now writing the second last term]

$= \left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \ldots \ldots \times \left\{1 + \frac{1}{k^1}\right\} \times \{1 + \frac{1}{(k+1)^1}\}$

$= (k + 1)^1 \times \left\{1 + \frac{1}{(k+1)^1}\right\}$ [Using 1]

$= (k + 1)^1 \times \left\{\frac{(k+1)+1}{(k+1)^1}\right\}$

$= (k + 1)^2 \times \left\{\frac{(k+2)^1}{(k+1)^2}\right\}$

= k + 2

= RHS

LHS = RHS

Therefore, P (k + 1) is true whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for

Where n is a natural number

Hence proved.

Q. 16. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

n × (n + 1) × (n + 2) is multiple of 6

Answer : To Prove:

n × (n + 1) × (n + 2) is multiple of 6

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

n × (n + 1) × (n + 2) is multiple of 6

Let P(n): $n \times (n + 1) \times (n + 2)$, which is multiple of 6

For n = 1 P(n) is true since $1 \times (1 + 1) \times (1 + 2) = 6$, which is multiple of 6

Assume P(k) is true for some positive integer k , ie,

= k × (k + 1) × (k + 2) = 6m, where m ∈ N …(1)

We will now prove that P(k + 1) is true whenever P(k) is true

Consider,

= (k + 1) × ((k + 1) + 1) × ((k + 1) + 2)

= (k + 1)×{ k + 2 }×{ (k + 2) + 1}

= [(k + 1)×(k + 2)× (k + 2)] + (k + 1)×(k + 2)

= [k×(k + 1)×(k + 2) + 2×(k + 1)×(k + 2)] + (k + 1)×(k + 2)

= [6m + 2×(k + 1)×(k + 2)] + (k + 1)×(k + 2)

= 6m + 3×(k + 1)×(k + 2)

Now, (k + 1) & (k + 2) are consecutive integers, so their product is even

Then, (k + 1) × (k + 2) = 2×w (even)

Therefore,

= 6m + 3 × [2×w]

= 6m + 6×w

= 6(m + w)

= 6×q where q = (m + w) is some natural number

Therefore

$(k + 1) \times ((k + 1) + 1) \times ((k + 1) + 2)$ is multiple of 6

Therefore, P (k + 1) is true whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for all natural numbers, ie, N

Hence proved.

Q. 17. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

($x^{2n} - y^{2n}$) is divisible by (x + y).

Answer : To Prove:

$x^{2n} - y^{2n}$ *is divisible by* $x + y$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

Let P(n): $x^{2n} - y^{2n}$ *is divisible by* $x + y$

For n = 1 P(n) is true since $x^{2n} - y^{2n} = x^2 - y^2 = (x + y) \times (x - y)$

Which is divisible by x + y

Assume P(k) is true for some positive integer k , ie,

= $x^{2k} - y^{2k}$ *is divisible by* $x + y$

Let $x^{2k} - y^{2k} = m \times (x + y)$, where m ∈ N …(1)

We will now prove that P(k + 1) is true whenever P(k) is true

Consider,

$= x^{2(k+1)} - y^{2(k+1)}$

$= x^{2k} \times x^2 - y^{2k} \times y^2$

$= x^2(x^{2k} - y^{2k} + y^{2k}) - y^{2k} \times y^2$ [Adding and subtracting y^{2k}]

$= x^2(\mathrm{m} \times (\mathrm{x} + \mathrm{y}) + y^{2k}) - y^{2k} \times y^2$ [Using 1]

$= m \times (x + y)x^2 + y^{2k} x^2 - y^{2k} y^2$

$= m \times (x + y)x^2 + y^{2k} (x^2 - y^2)$

$= m \times (x + y)x^2 + y^{2k} (x - y)(x + y)$

$= (x + y) \{ mx^2 + y^{2k} (x - y) \}$, which is factor of (x + y)

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for all natural numbers ie, N

Hence proved

Q. 18. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

(x^{2n} – 1) - 1 is divisible by (x – y), where x ≠ 1.

Answer : To Prove:

$x^{2n-1} - 1 \; is \; divisible \; by \; x - 1$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

Let P(n): $x^{2n-1} - 1 \; is \; divisible \; by \; x - 1$

For n = 1

P(n) is true since $x^{2n-1} - 1 = x^{2-1} - 1 = (x - 1)$

Which is divisible by x - 1

Assume P(k) is true for some positive integer k , ie,

= $x^{2k-1} - 1$ *is divisible by* $x - 1$

$Let\ x^{2k-1} - 1 = m \times (x-1)$, where m ∈ N …(1)

We will now prove that P(k + 1) is true whenever P(k) is true

Consider,

$= x^{2(k+1)-1} - 1$

$= x^{2k-1} \times x^2 - 1$

$= x^2(x^{2k-1}) - 1$

$= x^2(x^{2k-1} - 1 + 1) - 1$ [Adding and subtracting 1]

$= x^2(\mathrm{m} \times (\mathrm{x}-1) + 1) - 1$ [Using 1]

$= x^2(\mathrm{m} \times (\mathrm{x}-1)) + x^2 \times 1 - 1$

$= x^2(\mathrm{m} \times (\mathrm{x}-1)) + x^2 - 1$

$= x^2(\mathrm{m} \times (\mathrm{x}-1)) + (x^1 - 1)(\mathrm{x} + 1)$

$= (x-1)\{mx^2 + (x+1)\}$, which is factor of (x - 1)

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for all natural numbers, ie, N.

Hence proved.

Q. 19. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$\{(41)^n - (14)^n\}$ is divisible by 27.

Answer : To Prove:

$41^n - 14^n$ *is a divisible of* 27

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

Let P(n): $41^n - 14^n \text{ is a divisible of } 27$

For n = 1 P(n) is true since $41^n - 14^n = 41^1 - 14^1 = 27$

Which is multiple of 27

Assume P(k) is true for some positive integer k , ie,

= $41^n - 14^n \text{ is a divisible of } 27$

$\therefore\ 41^k - 14^k = m \times 27$, where m ∈ N …(1)

We will now prove that P(k + 1) is true whenever P(k) is true

Consider,

$= 41^{k+1} - 14^{k+1}$

$= 41^k \times 41 - 14^k \times 14$

= $41(41^k - 14^k + 14^k) - 14^k \times 14$ [Adding and subtracting 14^k]

$= 41(41^k - 14^k) + 41 \times 14^k - 14^k \times 14$

$= 41(27\text{m}) + 14^k(41 - 14)$ [Using 1]

$= 41(27\text{m}) + 14^k(27)$

$= 27(41m + 14^k)$

$= 27 \times r$, where r = $(41m + 14^k)$ is a natural number

Therefore $41^{k+1} - 14^{k+1}$ is divisible of 27

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for all natural numbers, ie, N.

Hence proved.

Q. 20. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

(4^n + 15n – 1) is divisible by 9.

Answer : To Prove:

$4^n + 15n - 1 \; is\; a\; divisible\; of\; 9$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

Let P(n): $4^n + 15n - 1 \; is\; a\; divisible\; of\; 9$

For n = 1 P(n) is true since $4^n + 15n - 1 = 4^1 + 15 \times 1 - 1 = 18$

Which is divisible of 9

Assume P(k) is true for some positive integer k , ie,

= $4^k + 15k - 1 \; is\; a\; divisible\; of\; 9$

$\therefore\; 4^k + 15k - 1 = m \times 9$, where m ∈ N …(1)

We will now prove that P(k + 1) is true whenever P(k) is true.

Consider,

$= 4^{k+1} + 15(k + 1) - 1$

$= 4^k \times 4 + 15k + 15 - 1$

$= 4^k \times 4 + 15k + 14 + (60k + 4) - (60k + 4)$ [Adding and subtracting

$60k + 4$]

$= (4^{k+1} + 60k - 4) + 15k + 14 - (60k - 4)$

$= 4(4^k + 15k - 1) + 15k + 14 - (60k - 4)$

$= 4(9\text{m}) - 45k + 18$ [Using 1]

$= 4(9\text{m}) - 9(5k - 2)$

$= 9\,[\,(4m) - (5k - 2)]$

$= 9 \times r$, where r = $[\,(4m) - (5k - 2)]$ is a natural number

Therefore $4^k + 15k - 1\ is\ a\ divisible\ of\ 9$

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for all natural numbers, ie, N.

Hence proved.

Q. 21. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

(3^{2n+2} – 8n – 9) is divisible by 8.

Answer : To Prove:

$3^{2n+2} - 8n - 9\ is\ a\ divisible\ of\ 8$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

Let P(n): $3^{2n+2} - 8n - 9\ is\ a\ divisible\ of\ 8$

For n = 1 P(n) is true since

$3^{2n+2} - 8n - 9 = 3^{2+2} - 8 \times 1 - 9 = 81 - 17 = 64$

Which is divisible of 8

Assume P(k) is true for some positive integer k , ie,

= $3^{2k+2} - 8k - 9\ is\ a\ divisible\ of\ 8$

$\therefore\ 3^{2k+2} - 8k - 9 = m \times 8$, where m ∈ N …(1)

We will now prove that P(k + 1) is true whenever P(k) is true

Consider,

$= 3^{2(k+1)+2} - 8(k + 1) - 9$

$= 3^{2(k+1)} \times 3^2 - 8k - 8 - 9$

$= 3^2(3^{2(k+1)} - 8k - 9 + 8k + 9) - 8k - 8 - 9$

[Adding and subtracting 8k + 9]

$= 3^2(3^{2(k+1)} - 8k - 9) + 3^2(8k + 9) - 8k - 17$

$= 9(3^{2k+2} - 8k - 9) + 9(8k + 9) - 8k - 17$

= 9(8m) + 72k + 81 -8k-17 [Using 1]

= 9(8m) + 64k + 64

= 8(9m + 8k + 8)

= 8×r, where r = 9m + 8k + 8 is a natural number

Therefore $3^{2k+2} - 8k - 9$ is a divisible of 8

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for all natural numbers, ie, N.

Hence proved.

Q. 22. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

(2^{3n} – 1) is a multiple of 7

Answer : To Prove:

$2^{3n} - 1$, which is multiple of 7

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

$2^{3n} - 1$ is multiple of 7

Let P(n): $2^{3n} - 1$, which is multiple of 7

For n = 1 P(n) is true since $2^3 - 1 = 8 - 1 = 7$, which is multiple of 7

Assume P(k) is true for some positive integer k , ie,

= $2^{3k} - 1 = 7m$, where m ∈ N …(1)

We will now prove that P(k + 1) is true whenever P(k) is true

Consider,

$= 2^{3(k+1)} - 1$

$= 2^{3k} \times 2^3 - 1$

$= 2^{3k} \times 2^3 + 2^3 - 2^3 - 1$ [Adding and subtracting 2^3]

$= 2^3(2^{3k} - 1) + 2^3 - 1$

$= 2^3(7m) + 2^3 - 1$ [Using 1]

$= 2^3(7m) + 7$

$= 7\,(\,2^3 m + 1)$

$= 7 \times r$, where r = $2^3 m + 1$ is a natural number

Therefore $2^{3n} - 1$ is multiple of 7

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for all natural numbers ie, N

Hence proved

Q. 23. Using the principle of mathematical induction, prove each of the following for all n ϵ N:

$3^n \geq 2^n$.

Answer: To Prove:

$3^n \geq 2^n$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

Let P(n): $3^n \geq 2^n$

For n = 1 P(n) is true since $3^n \geq 2^n \; i \times e \times \; 3 \geq 2$, which is true

Assume P(k) is true for some positive integer k , ie,

= $3^k \geq 2^k$ …(1)

We will now prove that P(k + 1) is true whenever P(k) is true

Consider,

$= \; 3^{(k+1)}$

$\therefore \; 3^{(k+1)} \; = \; 3^k \times 3 > \; 2^k \times 3$ [Using 1]

$= \; 3^k \times 3 > \; 2^k \times 2 \times \frac{3}{2}$ [Multiplying and dividing by 2 on RHS]

$= \; 3^{k+1} \; > \; 2^{k+1} \times \frac{3}{2}$

Now, $2^{k+1} \times \frac{3}{2} \; > \; 2^{k+1}$

$\therefore 3^{k+1} \; > \; 2^{k+1}$

Therefore, P (k + 1) is true whenever P(k) is true

By the principle of mathematical induction, P(n) is true for all natural numbers, ie, N.

Hence proved.