Some Special Series

Exercise 13A

Q. 1. Find the sum of the series whose nth term is given by:

($3n^2$ + 2n)

Answer : It is given in the question that the n^{th} term of the series,

$a_n = 3n^2 + 2n$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (3n^2 + 2n)$$

$$= \sum_{n=1}^{n} (3n^2) + \sum_{n=1}^{n} (2n)$$

$$= 3\sum_{n=1}^{n} (n^2) + 2\sum_{n=1}^{n} (n)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = 3\sum_{n=1}^{n}(n^2) + 2\sum_{n=1}^{n}(n)$$

From the above identities,

$$S_n = 3\sum_{n=1}^{n}(n^2) + 2\sum_{n=1}^{n}(n)$$

$$S_n = 3\left(\frac{n(n+1)(2n+1)}{6}\right) + 2\left(\frac{n(n+1)}{2}\right)$$

$$= \left(\frac{n(n+1)(2n+1)}{2}\right) + n(n+1)$$

$$= n(n+1)\left(\frac{2n+1}{2} + 1\right)$$

$$= \frac{n}{2}(n+1)(2n+3)$$

Hence, Sum of the series, $S_n = \frac{n}{2}(n+1)(2n+3)$

Q. 2. Find the sum of the series whose nth term is given by:

n (n + 1) (n + 4)

Answer : It is given in the question that the n^{th} term of the series,

$a_n = n (n + 1) (n + 4)$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (n(n+1)(n+4))$$

$= \sum_{n=1}^{n} (n^3 + 5n^2 + 4n)$

$= \sum_{n=1}^{n} (n^3) + 5\sum_{n=1}^{n} (n^2) + 4\sum_{n=1}^{n} (n)$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \dots n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \dots n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n}(n^3) + 5\sum_{n=1}^{n}(n^2) + 4\sum_{n=1}^{n}(n)$$

From the above identities,

$$S_n = \sum_{n=1}^{n}(n^3) + 5\sum_{n=1}^{n}(n^2) + 4\sum_{n=1}^{n}(n)$$

$$S_n = \left(\frac{n(n+1)}{2}\right)^2 + 5\left(\frac{n(n+1)(2n+1)}{6}\right) + 4\left(\frac{n(n+1)}{2}\right)$$

$$S_n = \frac{\left(n(n+1)\right)^2}{4} + 5\left(\frac{n(n+1)(2n+1)}{6}\right) + 2(n(n+1))$$

$$= n(n+1)\left(\frac{n(n+1)}{4} + 5\left(\frac{2n+1}{6}\right) + 2\right)$$

$$= \frac{n(n+1)}{24}(6n(n+1) + 20(2n+1) + 48)$$

$$= \frac{n(n+1)}{24}(6n^2 + 46n + 68)$$

$$= \frac{n(n+1)}{12}(63 + 23n + 34)$$

Hence, the Sum of the series, $S_n = \frac{n(n+1)}{12}(63 + 23n + 34)$

Q. 3. Find the sum of the series whose nth term is given by:

$(4n^3 + 6n^2 + 2n)$

Answer : It is given in the question that the n^{th} term of the series,

$a_n = 4n^3 + 6n^2 + 2n$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (4n^3 + 6n^2 + 2n)$$

$$= \sum_{n=1}^{n} (4n^3) + \sum_{n=1}^{n} (6n^2) + \sum_{n=1}^{n} (2n)$$

$$= 4\sum_{n=1}^{n} (n^3) + 6\sum_{n=1}^{n} (n^2) + 2\sum_{n=1}^{n} (n)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = 4\sum_{n=1}^{n}(n^3) + 6\sum_{n=1}^{n}(n^2) + 2\sum_{n=1}^{n}(n)$$

From the above identities,

$$S_n = 4\sum_{n=1}^{n}(n^3) + 6\sum_{n=1}^{n}(n^2) + 2\sum_{n=1}^{n}(n)$$

$$S_n = 4\left(\frac{n(n+1)}{2}\right)^2 + 6\left(\frac{n(n+1)(2n+1)}{6}\right) + 2\left(\frac{n(n+1)}{2}\right)$$

$$= (n(n+1))^2 + n(n+1)(2n+1) + n(n+1)$$

$$= n(n+1)[n(n+1)(2n+1) + 1]$$

$$= n(n+1)[n^2 + 3n + 2]$$

$$= n(n+1)^2(n+2)$$

Hence, the Sum of the series, $S_n = n(n+1)^2(n+2)$

Q. 4. Find the sum of the series whose nth term is given by:

($3n^2$ – 3n + 2)

Answer : It is given in the question that the n^{th} term of the series,

$a_n = 3n^2 - 3n + 2$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n}(3n^2 - 3n + 2)$$

$$= \sum_{n=1}^{n} (3n^2) - \sum_{n=1}^{n} (3n) + \sum_{n=1}^{n} (2)$$

$$= 3\sum_{n=1}^{n} (n^2) - 3\sum_{n=1}^{n} (n) + \sum_{n=1}^{n} (2)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots .. n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = 3\sum_{n=1}^{n} (n^2) - 3\sum_{n=1}^{n} (n) + \sum_{n=1}^{n} (2)$$

$$S_n = 3\left(\frac{n(n+1)(2n+1)}{6}\right) - 3\left(\frac{n(n+1)}{2}\right) + 2n$$

$$= \left(\frac{n(n+1)(2n+1) \; - 3n(n+1) + 4n}{2}\right)$$

On simplifying,

$$S_n = n(n^2 + 1)$$

Hence, the sum of the series, $S_n = n(n^2 + 1)$

Q. 5. Find the sum of the series whose nth term is given by:

$(2n^2 – 3n + 5)$

Answer : It is given in the question that the n^{th} term of the series,

$a_n = 2n^2 – 3n + 5$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (2n^2 - 3n + 5)$$

$$= \sum_{n=1}^{n} (2n^2) - \sum_{n=1}^{n} (3n) + \sum_{n=1}^{n} (5)$$

$$= 2\sum_{n=1}^{n} (n^2) - 3\sum_{n=1}^{n} (n) + \sum_{n=1}^{n} (5)$$

__Note:__

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = 2\sum_{n=1}^{n}(n^2) - 3\sum_{n=1}^{n}(n) + \sum_{n=1}^{n}(5)$$

$$S_n = 2\left(\frac{n(n+1)(2n+1)}{6}\right) - 3\left(\frac{n(n+1)}{2}\right) + 5n$$

$$= \left(\frac{2n(n+1)(2n+1) - 9n(n+1) + 30n}{6}\right)$$

$$= \left(\frac{4n^3 - 3n^2 + 23n}{6}\right)$$

$$= \frac{n}{6}(4n^2 - 3n + 23)$$

Hence, the sum of the series, $S_n = \frac{n}{6}(4n^2 - 3n + 23)$

Q. 6. Find the sum of the series whose nth term is given by:

$(n^3 - 3^n)$

Answer : It is given in the question that the n^{th} term of the series,

$a_n = n^3 - 3^n$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (n^3 - 3^n)$$

$$= \sum_{n=1}^{n} (n^3) + \sum_{n=1}^{n} (3^n)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n}(n^3) + \sum_{n=1}^{n}(3^n)$$

$$S_n = \left(\frac{n(n+1)}{2}\right)^2 + \sum_{n=1}^{n}(3^n) \rightarrow (1)$$

The second term in the equation, $\sum_{n=1}^{n}(3^n) = 3^1 + 3^2 + \cdots 3^n$,

Forms a GP, with the common ratio, r = 3.

Sum of n terms of a GP, a, ar, ar^2, ar^3…ar^n.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Here, a= 3, r = 3;

So,

$$S_n = \frac{3(3^n-1)}{3-1} = \frac{3(3^n-1)}{2} \rightarrow (2)$$

Substitute (2) in (1);

$$S_n = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3(3^n - 1)}{2}$$

Hence, the sum of the series, $S_n = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3(3^n-1)}{2}$

Q. 7. Find the sum of the series:

($2^2 + 4^2 + 6^2 + 8^2 + \ldots$ to n terms)

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is … $2^2 + 4^2 + 6^2 + 8^2 + \ldots$ to n terms.

The series can be written as, $[(2 \times 1)^2, (2 \times 2)^2,$

$(2 \times 3)^2 \ldots (2 \times n)^2]$.

So, n^{th} term of the series,

$a_n = (2n)^2 = 4n^2$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (4n^2)$$

$$= 4\sum_{n=1}^{n} (n^2)$$

Note:

I. Sum of first n natural numbers, $1 + 2 + 3 + \ldots n$,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = 4\sum_{n=1}^{n}(n^2)$$

From, the above identities,

$$S_n = 4\sum_{n=1}^{n}(n^2)$$

$$S_n = 4\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{2}{3}[n(n+1)(2n+1)]$$

$$S_n = \frac{2}{3}[n(n+1)(2n+1)]$$

So, Sum of the series, $S_n = \frac{2}{3}[n(n+1)(2n+1)]$

Q. 8. Find the sum of the series:

($2^3 + 4^3 + 6^3 + 8^3 + \ldots$ to n terms)

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is … $2^3 + 4^3 + 6^3 + 8^3 + \ldots$ to n terms.

The series can be written as, $[(2 \times 1)^3, (2 \times 2)^3,$

$(2 \times 3)^3 \ldots (2 \times n)^3]$.

So, n^{th} term of the series,

$a_n = (2n)^3 = 8n^3$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (8n^3)$$

$$= 8\sum_{n=1}^{n} (n^3)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = 8\sum_{n=1}^{n}(n^3)$$

From, the above identities,

$$S_n = 8\sum_{n=1}^{n}(n^3)$$

$$S_n = 8\left(\frac{n(n+1)}{2}\right)^2$$

$$= 2[n(n+1)]^2$$

So, Sum of the series, $S_n = 2[n(n+1)]^2$

Q. 9. Find the sum of the series:

$(5^2 + 6^2 + 7^2 + \ldots + 20^2)$

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is $5^2, 6^2, 7^2 \ldots 20^2$.

The series can be written as, $[(1 + 4)^2, (2 + 4)^2, (3 + 4)^2 \ldots (16 + 4)^2]$.

So, n^{th} term of the series,

$a_n = (n + 4)^2$

With n = 16,

$a_n = n^2 + 8n + 16$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (n^2 + 8n + 16)$$

$$= \sum_{n=1}^{n} (n^2) + \sum_{n=1}^{n} (8n) + \sum_{n=1}^{n} (16)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n} (n^2) + \sum_{n=1}^{n} (8n) + \sum_{n=1}^{n} (16)$$

From, the above identities,

$$S_n = \sum_{n=1}^{n}(n^2) + \sum_{n=1}^{n}(8n) + \sum_{n=1}^{n}(16)$$

$$S_n = \left(\frac{n(n+1)(2n+1)}{6}\right) + 8\left(\frac{n(n+1)}{2}\right) + 16n$$

Here, n = 16 (from the question);

$$S_n = \left(\frac{(16)(17)(33)}{6}\right) + \left(\frac{(8)(16)(17)}{2}\right) + 16 \times 16$$

$S_n = 2840$

So, Sum of the series, $S_n = 2840$.

Q. 10. Find the sum of the series:

(1 × 2) + (2 × 3) + (3 × 4) + (4 × 5) + … to n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is (1 × 2) + (2 × 3) + (3 × 4) + (4 × 5) + … to n terms.

The series can be written as, [(1 x (1 + 1)), (2 x (2 + 1)),

(3 x (3 + 1)),…(n x (n + 1))].

So, n^{th} term of the series,

$a_n = n(n + 1)$

$a_n = n^2 + n$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (n^2 + n)$$

$$= \sum_{n=1}^{n} (n^2) + \sum_{n=1}^{n} (n)$$

Note:

V. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

VI. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

VII. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots .. n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

VIII. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n} (n^2) + \sum_{n=1}^{n} (n)$$

From, the above identities,

$$S_n = \sum_{n=1}^{n}(n^2) + \sum_{n=1}^{n}(n)$$

$$S_n = \left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)\left(\frac{2n+4}{3}\right)$$

$$S_n = \left(\frac{n(n+1)}{2}\right)\left(\frac{2n+4}{3}\right)$$

So, Sum of the series, $S_n = \left(\frac{n(n+1)(n+2)}{3}\right)$

Q. 11. Find the sum of the series:

(3 × 8) + (6 × 11) + (9 × 14) + … to n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is (3 × 8) + (6 × 11) + (9 × 14) + … to n terms.

The series can be written as, [(3 x 1) x (3 x 1 + 5)), (3 x 2) x (3 x 2 + 5))… (3n x (3n + 5))].

So, n^{th} term of the series,

$a_n = 3n\ (3n + 5)$

$a_n = 9n^2 + 15n$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n}(9n^2 + 15n)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots .. n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = 9\sum_{n=1}^{n}(n^2) + \sum_{n=1}^{n}(n)$$

From, the above identities,

$$S_n = \sum_{n=1}^{n}(9n^2) + \sum_{n=1}^{n}(15n)$$

$$S_n = 9\left(\frac{n(n+1)(2n+1)}{6}\right) + 15\left(\frac{n(n+1)}{2}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)(6n + 18)$$

$$S_n = 3n(n+1)(n+3)$$

So, Sum of the series, $S_n = 3n(n+1)(n+3)$

Q. 12. Find the sum of the series:

$(1 \times 2^2) + (2 \times 3^2) + (3 \times 4^2) + \ldots$ to n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is $(1 \times 2^2) + (2 \times 3^2) + (3 \times 4^2) + \ldots$ to n terms.

The series can be written as, $[(1 \times (1 + 1)^2), (2 \times (2 + 1)^2 \ldots (n \times (n + 1)^2]$.

So, n^{th} term of the series,

$a_n = n\,(n + 1)^2$

$a_n = n^3 + 2n^2 + n$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (n^3 + 2n^2 + n)$$

<u>**Note:**</u>

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n}(n^3) + 2\sum_{n=1}^{n}(n^2) + \sum_{n=1}^{n}(n)$$

From, the above identities,

$$S_n = \sum_{n=1}^{n}(n^3) + 2\sum_{n=1}^{n}(n^2) + \sum_{n=1}^{n}(n)$$

$$S_n = \left(\frac{n(n+1)}{2}\right)^2 + 2\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)\left[\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right]$$

$$S_n = \left(\frac{n(n+1)}{2}\right)\left[\frac{3n^2 + 11n + 10}{6}\right]$$

So, Sum of the series, $S_n = \left(\frac{n(n+1)}{12}\right)(3n^2 + 11n + 10)$

Q. 13. Find the sum of the series:

$(1 \times 2^2) + (3 \times 3^2) + (5 \times 4^2) + \ldots$ to n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is $(1 \times 2^2) + (3 \times 3^2) + (5 \times 4^2) + \ldots$ to n terms.

The series can be written as, $[(1 \times (1 + 1)^2), (2 \times (2 + 1)^2 \ldots (2n\text{-}1 \times (n + 1)^2]$.

So, n^{th} term of the series,

$a_n = (2n – 1)(n + 1)^2$

$= (2n – 1)(n^2 + 2n + 1)$

$=2n^3 + 3n^2 - 1$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (2n^3 + 3n^2 - 1)$$

<u>Note:</u>

I. Sum of first n natural numbers, $1 + 2 + 3 + \ldots n$,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = 2\sum_{n=1}^{n}(n^3) + 3\sum_{n=1}^{n}(n^2) - \sum_{n=1}^{n}(1)$$

From, the above identities,

$$S_n = 2\sum_{n=1}^{n}(n^3) + 3\sum_{n=1}^{n}(n^2) - \sum_{n=1}^{n}(1)$$

$$S_n = 2\left(\frac{n(n+1)}{2}\right)^2 + 3\left(\frac{n(n+1)(2n+1)}{6}\right) - n$$

$$= \left(\frac{n(n+1)}{2}\right)[n(n+1) + (2n+1)] - n$$

$$S_n = \left(\frac{n(n+1)}{2}\right)[n^2 + 3n + 1] - n$$

$$S_n = \left(\frac{n}{2}\right)[(n+1)(n^2 + 3n + 1) - 2]$$

$$S_n = \left(\frac{n}{2}\right)[n^3 + 4n^2 + 4n - 1]$$

So, Sum of the series, $S_n = \left(\frac{n}{2}\right)[n^3 + 4n^2 + 4n - 1]$

Q. 14. Find the sum of the series:

$(3 \times 1^2) + (5 \times 2^2) + (7 \times 3^2) + \ldots$ to n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is $(3 \times 1^2) + (5 \times 2^2) + (7 \times 3^2) + \ldots$to n terms.

The series can be written as, $[(3 \times 1^2), (5 \times 2^2 \ldots ((2n + 1) \times n^2]$.

So, n^{th} term of the series,

$a_n = (2n + 1)\, n^2$

$a_n = 2n^3 + n^2$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (2n^3 + n^2)$$

__**Note:**__

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n}(2n^3) + \sum_{n=1}^{n}(n^2)$$

From, the above identities,

$$S_n = 2\sum_{n=1}^{n}(n^3) + \sum_{n=1}^{n}(n^2)$$

$$S_n = 2\left(\frac{n(n+1)}{2}\right)^2 + \left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)\left[n(n+1) + \frac{(2n+1)}{3}\right]$$

$$S_n = \left(\frac{n(n+1)}{2}\right)\left[\frac{3n^2+5n+1}{3}\right]$$

So, Sum of the series, $S_n = \left(\frac{n(n+1)}{6}\right)(3n^2+5n+1)$

Q. 15. Find the sum of the series:

(1 × 2 × 3) + (2 × 3 × 4) + (3 × 4 × 5) + … to n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is (1 × 2 x 3) + (2 × 3 x 4) + (3 × 4 x 5) + … to n terms.

The series can be written as, [(1 x (1 + 1) x (1 + 2)), (2 x (2 + 1) x (2 + 2) … (n x (n + 1) x (n + 2)].

So, n^{th} term of the series,

$a_n = n (n + 1) (n + 2)$

$a_n = n^3 + 3n^2 + 2n$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (n^3 + 3n^2 + 2n)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n}(n^3) + \sum_{n=1}^{n}(3n^2) + \sum_{n=1}^{n}(2n)$$

From, the above identities,

$$S_n = \sum_{n=1}^{n}(n^3) + 3\sum_{n=1}^{n}(n^2) + 2\sum_{n=1}^{n}(n)$$

$$S_n = \left(\frac{n(n+1)}{2}\right)^2 + 3\left(\frac{n(n+1)(2n+1)}{6}\right) + 2\left(\frac{n(n+1)}{2}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)\left[\frac{n(n+1)}{2} + (2n+1) + 2\right]$$

$$= \left(\frac{n(n+1)}{2}\right)\left[\frac{n^2+5n+6}{2}\right]$$

$$= \left(\frac{n(n+1)}{2}\right)\left[\frac{(n+3)(n+2)}{2}\right]$$

So, Sum of the series, $S_n = \left(\frac{n(n+1)(n+2)(n+3)}{4}\right)$

Q. 16. Find the sum of the series:

(1 × 2 × 4) + (2 × 3 × 7) + (3 × 4 × 10) + … to n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is (1 × 2 × 4) + (2 × 3 × 7) + (3 × 4 × 10) + … to n terms.

The series can be written as, [(1 x (1 + 1) x (3 x 1 + 1)), (2 x (2 + 1) x (3 x 2 + 1))… (n x (n + 1) x (3 x n + 1)].

So, n^{th} term of the series,

$a_n = n (n + 1) (3n + 1)$

$a_n = 3n^3 + 4n^2 + n$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (3n^3 + 4n^2 + n)$$

<u>Note:</u>

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n}(n^3) + 2\sum_{n=1}^{n}(n^2) + \sum_{n=1}^{n}(n)$$

From, the above identities,

$$S_n = 3\sum_{n=1}^{n}(n^3) + 4\sum_{n=1}^{n}(n^2) + \sum_{n=1}^{n}(n)$$

$$S_n = 3\left(\frac{n(n+1)}{2}\right)^2 + 4\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)\left[\frac{3n(n+1)}{2} + \frac{4(2n+1)}{3} + 1\right]$$

$$S_n = \left(\frac{n(n+1)}{2}\right)\left[\frac{9n^2 + 25n + 14}{6}\right]$$

So, Sum of the series, $S_n = \left(\frac{n(n+1)}{12}\right)(9n^2 + 25n + 14)$

Q. 17. Find the sum of the series:

$\frac{1}{1x\,2} + \frac{1}{2x\,3} + \frac{1}{3x\,4} + ..$ **…. To n terms**

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the nth term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+..$….to n terms.

The series can be written as, $\frac{1}{1\times 2},\frac{1}{2\times 3},\frac{1}{3\times 4},..$…$\frac{1}{n\times(n+1)}$.

So, nth term of the series,

$$a_n = \frac{1}{n(n+1)}$$

By the method of partial fractions, we can factorize the above term.

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$a_1 = 1 - \frac{1}{2} \rightarrow (1)$$

$$a_2 = \frac{1}{2} - \frac{1}{3} \rightarrow (2)$$

$$a_{n-1} = \frac{1}{n-1} - \frac{1}{n} \rightarrow (n\text{-}1)^{th}\text{ equation}$$

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \rightarrow n^{th}\text{ equation}$$

Now, we need to find the sum of this series, S_n.

This can be found out by adding the equation (1), (2)…up to nth term.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

So, Sum of the series, $S_n = \frac{n}{n+1}$

Q. 18. Find the sum of the series:

$$\frac{1}{1x2}+\frac{1}{2x3}+\frac{1}{3x4}+\ldots\frac{1}{(2n-1)(2n+1)}$$

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is $\frac{1}{1\,x\,2}+\frac{1}{2\,x\,3}+\frac{1}{3\,x\,4}$ … to n terms.

So, n^{th} term of the series,

$$a_n = \frac{1}{(2n-1)(2n+1)}$$

By the method of partial fractions, we can factorize the above term.

$$a_n = \frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$1 = A(2n-1) + B(2n+1)$$

On equating the like term on RHS and LHS,

2A + 2B = 0 → (a)

A + B = 1 → (b)

On solving, we will get; $A = \frac{1}{2}; B = -\frac{1}{2}$

$$a_n = \frac{1}{(2n-1)(2n+1)} = \frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$a_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right) \rightarrow (1)$$

$$a_2 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) \rightarrow (2)$$

$$a_{n-1} = \frac{1}{2}\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) \rightarrow (n\text{-}1)^{\text{th}} \text{ equation}$$

$$a_n = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \rightarrow n^{\text{th}} \text{ equation}$$

Now, we need to find the sum of this series, S_n.

This can be found out by adding the equation (1), (2)…up to n^{th} term.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \frac{1}{2}\left[1 - \frac{1}{2n+1}\right] = \frac{n}{2n+1}$$

So, Sum of the series, $S_n = \frac{n}{2n+1}$

Q. 19. Find the sum of the series:

$$\frac{1}{(1\times 6)} + \frac{1}{(6\times 11)} + \frac{1}{(11\times 16)} + \ldots.. + \frac{1}{(5n-4)(5n+1)}$$

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is $(1 \times 2^2) + (2 \times 3^2) + (3 \times 4^2) + \ldots$ to n terms.

The series can be written as, $[(1 \times (1 + 1)^2), (2 \times (2 + 1)^2 \ldots (n \times (n + 1)^2]$.

So, n^{th} term of the series,

$a_n = n (n + 1)^2$

$a_n = n^3 + 2n^2 + n$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (n^3 + 2n^2 + n)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n}(n^3) + 2\sum_{n=1}^{n}(n^2) + \sum_{n=1}^{n}(n)$$

From, the above identities,

$$S_n = \sum_{n=1}^{n}(n^3) + 2\sum_{n=1}^{n}(n^2) + \sum_{n=1}^{n}(n)$$

$$S_n = \left(\frac{n(n+1)}{2}\right)^2 + 2\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)\left[\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right]$$

$$S_n = \left(\frac{n(n+1)}{2}\right)\left[\frac{3n^2 + 11n + 10}{6}\right]$$

So, Sum of the series, $S_n = \left(\frac{n(n+1)}{12}\right)(3n^2 + 11n + 10)$

Q. 20. Find the sum of the series:

$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots$ **to n terms**

Answer : In the given question we need to find the sum of the series.

For that, firs, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \cdots$ to n terms

The series can be written as,

$\frac{1^3}{1}, \frac{1^3+2^3}{1+3}, \frac{1^3+2^3+3^3}{1+3+5}, \ldots \frac{1^3+2^3+3^3+\cdots+n^3}{1+3+5+\cdots+(2n-1)}$.

So, n^{th} term of the series,

$$a_n = \frac{1^3 + \ 2^3 + 3^3 + \cdots n^3.}{1 + 3 + 5 + \cdots + (2n - 1)}$$

The denominator of 'a_n' forms an AP with first term a = 1, last term = 2n-1 and common difference, d= 2.

Now, Sum of the AP, $T_n = \frac{n}{2}[2a + (n - 1)d]$

$$= \frac{n}{2}[2 + (n - 1)2] = n^2$$

$$a_n = \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^2} = \frac{(n+1)^2}{4}$$

$$a_n = \frac{n^2 + 2n + 1}{4}$$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} \left(\frac{n^2 + 2n + 1}{4}\right)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots.. n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \frac{1}{4}\left[\sum_{n=1}^{n}(n^2) + 2\sum_{n=1}^{n}(n) + \sum_{n=1}^{n}(1)\right]$$

From, the above identities,

$$S_n = \frac{1}{4}\left[\sum_{n=1}^{n}(n^2) + 2\sum_{n=1}^{n}(n) + \sum_{n=1}^{n}(1)\right]$$

$$S_n = \frac{1}{4}\left[\left(\frac{n(n+1)(2n+1)}{6}\right) + 2\left(\frac{n(n+1)}{2}\right) + n\right]$$

$$= \left(\frac{n}{4}\right)\left[\frac{(n+1)(2n+1)}{6} + (n+1) \; + 1\right]$$

$$S_n = \left(\frac{n}{4}\right)\left[\frac{2n^2 + 9n + 13}{6}\right]$$

So, Sum of the series, $S_n = \left(\frac{n}{24}\right)(2n^2 + 9n + 13)$

Q. 21. Find the sum of the series:

3 + 15 + 35 + 63 +...to n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is 3, 15, 35, 63 … to n terms.

The series can be written as, $[2^2 - 1, 4^2-1, 6^2-1 \ldots (2n)^2-1]$.

So, n^{th} term of the series,

$a_n = (2n)^2 - 1$

$a_n = 4n^2 - 1$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} (4n^2 - 1)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n} (4n^2) - \sum_{n=1}^{n} (1)$$

From, the above identities,

$$S_n = 4\sum_{n=1}^{n} (n^2) - \sum_{n=1}^{n} (1)$$

$$S_n = 4\left(\frac{n(n+1)(2n+1)}{6}\right) - (n)$$

$$= \left(\frac{n}{3}\right)[2(n+1)(2n+1) - 3]$$

$$S_n = \left(\frac{n}{3}\right)[4n^2 + 6n - 1]$$

So, Sum of the series, $S_n = \frac{n}{3}[4n^2 + 6n - 1]$

Q. 22. Find the sum of the series:

1 + 5 + 12 + 22 + 35 +... to n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is 1 + 5 + 12 + 22 + 35 … to n terms.

This question can be solved by the method of difference.

Note:

Consider a sequence a_1, a_2, a_3 …such that the Sequence $a_2 – a_1$, $a_3 – a_2$… is either an. A.P. or a G.P.

The n^{th} term, of this sequence, is obtained as follows:

$S = a_1 + a_2 + a_3 + … + a_{n–1} + a_n \rightarrow (1)$

$S = a_1 + a_2 + … + a_{n–2} + a_{n–1} + a_n \rightarrow (2)$

Subtracting (2) from (1),

We get, $a_n = a_1 + [(a_2–a_1) + (a_3–a_2) + … (a_n – a_{n–1})]$.

Since the terms within the brackets are either in an A.P. or a G.P, we can find the value of a_n the n^{th} term.

Thus, we can find the sum of the n terms of the sequence as,

$$S_n = \sum_{k=1}^{n} a_k$$

So,

By using the method of difference, we can find the n^{th} term of the expression.

$S_n = 1 + 5 + 12 + 22 + 35 + ….. + a_n \rightarrow (1)$

$S_n = 1 + 5 + 12 + 22 + 35 + …. + a_n \rightarrow (2)$

$(1) – (2) \rightarrow 0 = 1 + 4 + 7 + 10 + ….. - a_n$

So, n^{th} term of the series,

$a_n = 1 + 4 + 7 + 10 + ….$

So, the n^{th} term form an AP, with the first term, a = 1; common difference, d = 3.

The required n^{th} term of the series is the same as the sum of n terms of AP.

Sum of n terms of an AP, $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_n = \frac{n}{2}[2 \text{ x } 1 + 3(n-1)]$$

$$= \frac{n}{2}[3n-1] = \frac{3n^2-n}{2}$$

So, n^{th} term of the series, $a_n = \frac{3n^2-n}{2}$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} \frac{3n^2-n}{2}$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \frac{1}{2}\left[3\sum_{n=1}^{n}(n^2) - \sum_{n=1}^{n}(n)\right]$$

From, the above identities,

$$S_n = \frac{1}{2}\left[3\sum_{n=1}^{n}(n^2) - \sum_{n=1}^{n}(n)\right]$$

$$S_n = \frac{1}{2}\left[3\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{n(n+1)}{2}\right)\right]$$

$$= \left(\frac{n(n+1)}{4}\right)[(2n+1)-1]$$

$$S_n = \left(\frac{n^2(n+1)}{2}\right)$$

So, Sum of the series, $S_n = \left(\frac{n^2(n+1)}{2}\right)$

Q. 23. Find the sum of the series:

5 + 7 + 13 + 31 + 85 + …. To n terms

Answer : In the given question we need to find the sum of the series.

For that, first, we need to find the n^{th} term of the series so that we can use summation of the series with standard identities and get the required sum.

The series given is 5 + 7 + 13 + 31 + 85 + …. + n terms.

This question can be solved by the method of difference.

<u>Note:</u>

Consider a sequence a_1, a_2, a_3 …such that the Sequence $a_2 - a_1$, $a_3 - a_2$… is either an. A.P. or a G.P.

The n^{th} term, of this sequence, is obtained as follows:

$S = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \rightarrow (1)$

$S = a_1 + a_2 + \ldots + a_{n-2} + a_{n-1} + a_n \rightarrow (2)$

Subtracting (2) from (1),

We get, $a_n = a_1 + [(a_2 - a_1) + (a_3 - a_2) + \ldots (a_n - a_{n-1})]$.

Since the terms within the brackets are either in an A.P. or a G.P, we can find the value of a_n, the n^{th} term.

Thus, we can find the sum of the n terms of the sequence as,

$$S_n = \sum_{k=1}^{n} a_k$$

So,

By using the method of difference, we can find the n^{th} term of the expression.

$S_n = 5 + 7 + 13 + 31 + 85 + \ldots.. + a_n \rightarrow (1)$

$S_n = 5 + 7 + 13 + 31 + 85 + \ldots. + a_n \rightarrow (2)$

$(1) - (2) \rightarrow 0 = 5 + 2 + 6 + 18 + 54 + \ldots.. + (a_n - a_{n-1}) - a_n$

So, n^{th} term of the series,

$a_n = 5 + 2 + 6 + 18 + 54 + \ldots.$

In the resulting series obtained, starting from 2, 6, 18…forms a GP.

So, the n^{th} term forms a GP, with the first term, a = 2; common ratio, r = 3.

The required n^{th} term of the series is the same as the sum of n terms of GP and 5.

The GP is 2 + 6 + 18 + 54 +… (n-1) terms.

Sum of n terms of a GP, $S_n = \frac{a(r^n - 1)}{r-1}$

Sum of (n-1) terms of a GP, $S_n = \frac{a(r^{n-1} - 1)}{r-1}$

$$S_n = \frac{2(3^{n-1} - 1)}{2}$$

$$= 3^{n-1} - 1$$

$$= \frac{3^n}{3} - 1$$

So, n^{th} term of the series, $a_n = \frac{3^n}{3} - 1 + 5 = \frac{3^n}{3} + 4$

Now, we need to find the sum of this series, S_n.

$$S_n = \sum_{n=1}^{n} a_n$$

$$S_n = \sum_{n=1}^{n} \left(\frac{3^n}{3} + 4\right)$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So, for the given series, we need to find,

$$S_n = \sum_{n=1}^{n}\left(\frac{3^n}{3} + 4\right)$$

From, the above identities,

$S_n = \frac{1}{3}\sum_{n=1}^{n}(3^n) + \sum_{n=1}^{n} 4$ → (a)

The first term in (a) is a GP, with the first term, a = 3 and common ratio, r = 3.

Sum of n terms of GP, $S_n{}^1 = \frac{a(r^n - 1)}{r - 1}$

$$S_n{}^1 = \frac{3(3^n - 1)}{3 - 1} = \frac{3(3^n - 1)}{2}$$

$$S_n = \frac{1}{3}\left(\frac{3(3^n - 1)}{2}\right) + 4n$$

$$= \left(\frac{(3^n - 1)}{2}\right) + 4n$$

$$S_n = \left[\frac{8n + 3^n - 1}{2}\right]$$

So, Sum of the series, $S_n = \frac{n}{2}[8n + 3^n - 1]$

Q. 24. If $S_k = \frac{(1 + 2 + 3 + \ldots k)}{k}$ **, prove that ,**

$\left(S_1{}^2 + S_2{}^2 + \cdots S_n{}^2\right) = \frac{n}{24}(2n^2 + 9n + 13)$.

Answer : Given, $S_k = \frac{(1+2+3+\cdots k)}{k}$

To prove: $\left(S_1{}^2 + S_2{}^2 + \cdots S_n{}^2\right) = \frac{n}{24}(2n^2 + 9n + 13)$

$$(S_k) = \frac{1+2+3+\cdots k}{k}$$

Note:

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So,

$$(S_k) = \frac{k(k+1)}{2k} = \frac{k\ +1}{2}$$

Now, the Left hand side of the condition given in the question can be written as,

$$\left(S_1{}^2 + \ S_2{}^2 + \cdots S_n{}^2\right) = \sum_{k=1}^{n} S_k{}^2$$

The required LHS, $S_n = \sum_{k=1}^{n} S_k{}^2$

$$\sum_{k=1}^{n} S_k{}^2 = \sum_{k=1}^{n} \left(\frac{k+1}{2}\right)^2$$

$$= \sum_{k=1}^{n} \frac{k^2 \ + 2k + 1}{4}$$

$$= \frac{1}{4}\left(\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right)$$

$$S_n = \frac{1}{4}\left[\left(\frac{n(n+1)(2n+1)}{6}\right) + 2\left(\frac{n(n+1)}{2}\right) + n\right]$$

$$= \frac{1}{4}\left[n(n+1)\left(\left(\frac{(2n+1)}{6}\right) + 1\right) + n\right]$$

$$= \frac{n}{4}\left[\left(\left(\frac{(n+1)(2n+7)+6}{6}\right)\right)\right]$$

$$S_n = \frac{n}{24}\,[2n^2 + 9n + 13]$$

So,

$$(\,S_1{}^2 + \ S_2{}^2 + \cdots\ S_n{}^2\,) = \sum_{k=1}^{n} S_k{}^2 = \ S_n = \frac{n}{24}\,[2n^2 + 9n + 13]$$

$$(\,S_1{}^2 + \ S_2{}^2 + \cdots\ S_n{}^2\,) = \tfrac{n}{24}\,[2n^2 + 9n + 13]$$

With$S_k = \frac{(1+2+3+\cdots k)}{k}$,

Q. 25. If S_n denotes the sum of the cubes of the first n natural numbers and s_n denotes the sum of the first n natural numbers then find the value of $\sum_{k=1}^{n} \frac{S_k}{s_k}$.

Answer : Given in the question, S_n denotes the sum of the cubes of the first n natural numbers.

s_n denotes the sum of the first n natural numbers.

<u>Note:</u>

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

III. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots .. n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

IV. Sum of a constant k, N times,

$$\sum_{k=1}^{N} k = Nk$$

So,

S_n denotes the sum of the cubes of the first n natural numbers. (Given data in the question).

$$S_n = \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

s_n denotes the sum of the first n natural numbers.

$$s_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\frac{S_k}{s_k} = \frac{\left(\frac{k(k+1)}{2}\right)^2}{\frac{k(\ k+1)}{2}} = \frac{k(k+1)}{2}$$

To determine the given ratio in the question,

$$\sum_{k=1}^{n} \frac{S_k}{s_k} = \sum_{k=1}^{n} \frac{k(k+1)}{2}$$

$$= \frac{1}{2}\left[\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k\right]$$

$$= \frac{1}{2}\left[\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)\right]$$

$$= \frac{n(n+1)}{4}\left[\left(\frac{(2n+1)}{3}\right) + 1\right]$$

$$= \frac{n(n+1)}{4}\left[\left(\frac{(2n+4)}{3}\right)\right]$$

$$= \frac{n(n+1)(n+2)}{6}$$

So, the value of $\sum_{k=1}^{n} \frac{S_k}{s_k} = \frac{n(n+1)(n+2)}{6}$

Exercise 13B

Q. 1. Find the sum (2 + 4 + 6 + 8 +… + 100).

Answer : It is required to find the sum of (2 + 4 + 6 + 8 +… 100).

Now, consider the series (2 + 4 + 6 + 8 +… 100).

If we take a common factor of 2 from all the terms, then,

the series becomes,

2 (1 + 2 + 3 + 4 +… 50).

So, we need to find the sum of first 50 natural numbers.

Note:

Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

From the above identities,

So, Sum of first 50 natural numbers $= \frac{n(n+1)}{2}$

$$= \frac{50(51)}{2}$$

$$= 1275$$

(2 + 4 + 6 + 8 +… 100) = 2 (1 + 2 + 3 + 4 +… 50)

= 2 x 1275 = 2550

Q. 2. Find the sum (41 + 42 + 43 + …. + 100).

Answer : It is required to find the sum (41 + 42 + 43 + …. + 100).

(41 + 42 + 43 + …. + 100) = Sum of integers starting from 1 to 100 – Sum of integers starting from 1 to 40.

Note:

Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

From the above identities,

So, Sum of integers starting from 1 to 100 $= \frac{n(n+1)}{2}$

$$= \frac{100(101)}{2}$$

$$= 5050$$

So, Sum of integers starting from 1 to 40 $= \frac{n(n+1)}{2}$

$$= \frac{40(41)}{2}$$

$$= 820$$

(41 + 42 + 43 + …. + 100) = Sum of integers starting from 1 to 100 – Sum of integers starting from 1 to 40.

(41 + 42 + 43 + …. + 100) = 5050 – 820 = 4230

Q. 3. Find the sum $11^2 + 12^2 + 13^2 + \ldots 20^2$

Answer : It is required to find the sum $11^2 + 12^2 + 13^2 + \ldots 20^2$

$11^2 + 12^2 + 13^2 + \ldots 20^2$ = Sum of squares of natural numbers starting from 1 to 20 – Sum of squares of natural numbers starting from 1 to 10.

<u>Note:</u>

Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

From the above identities,

Sum of squares of natural numbers starting from 1 to 20

$$= \frac{20(21)(41)}{6} = 2870$$

Sum of squares of natural numbers starting from 1 to 10

$$= \frac{10(11)(21)}{6} = 385$$

$11^2 + 12^2 + 13^2 + \ldots 20^2$ = Sum of squares of natural numbers starting from 1 to 20 – Sum of squares of natural numbers starting from 1 to 10.

$11^2 + 12^2 + 13^2 + \ldots 20^2 = 2870 - 385 = 2485$

Q. 4. Find the sum $6^3 + 7^3 + 8^3 + 9^3 + 10^3$.

Answer : It is required to find the sum $6^3 + 7^3 + 8^3 + 9^3 + 10^3$.

$6^3 + 7^3 + 8^3 + 9^3 + 10^3$= Sum of cubes of natural numbers starting from 1 to 10 – Sum of cubes of natural numbers starting from 1 to 5.

Note:

Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

From the above identities,

Sum of cubes of natural numbers starting from 1 to 10

$$= \left(\frac{10(11)}{2}\right)^2 = 3025$$

Sum of cubes of natural numbers starting from 1 to 5

$$= \left(\frac{5(6)}{2}\right)^2 = 225$$

$6^3 + 7^3 + 8^3 + 9^3 + 10^3$ = Sum of cubes of natural numbers starting from 1 to 10 – Sum of cubes of natural numbers starting from 1 to 5.

$6^3 + 7^3 + 8^3 + 9^3 + 10^3 = 3025 - 225 = 2800$

Q. 5. If $\sum_{k=1}^{n} k = 210$, find the value of $\sum_{k=1}^{n} k^2$.

Answer : It is given that, $\sum_{k=1}^{n} k = 210$

<u>Note:</u>

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

From the above identities,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = 210$$

$n^2 + n = 420$

$n^2 + n - 420 = 0$

$(n - 20)(n + 21) = 0$

n = 20 or -21

Since, n is the number of integers, n = 20

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

So, $\sum_{k=1}^{n} k^2 = \frac{20(21)(41)}{6} = 2870$

Q. 6. If $\sum_{k=1}^{n} k = 45$, **find the value of** $\sum_{k=1}^{n} k^3$.

Answer : It is given that, $\sum_{k=1}^{n} k = 45$

<u>Note:</u>

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

II. Sum of cubes of first n natural numbers, $1^3 + 2^3 + 3^3 + \ldots . . n^3$,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

From the above identities,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = 45$$

We need to find,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = 45^2 = 2025$$

Q. 7. Find the sum of the series $\{2^2 + 4^2 + 6^2 + \ldots . + (2n)^2\}$

Answer : We need to find the sum of the series $\{2^2 + 4^2 + 6^2 + \ldots . + (2n)^2\}$.

So, we can find it by using summation of the n^{th} term of the given series.

The n^{th} term of the series is $(2n)^{2} = 4n^2$

(Given data)

$a_n = 4n^2$

Now, sum of the series, $S_n = \sum_{k=1}^{n} a_k$

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} 4n^2 = 4\sum_{k=1}^{n} n^2$$

Note:

I. Sum of squares of first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots . n^2$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_n = 4\sum_{k=1}^{n} n^2 = 4\,\frac{n(n+1)(2n+1)}{6}$$

$$S_n = \{22 + 42 + 62 + \ldots . + (2n)^2\}$$

$$= 4\,\frac{n(n+1)(2n+1)}{6}$$

$$S_n = \frac{2}{3}n(n+1)(2n+1)$$

Q. 8. Find the sum of 10 terms of the geometric series $\sqrt{2} + \sqrt{6} + \sqrt{18} + \ldots .$

Answer : We need to find the sum of 10 terms of GP.

Sum of n terms of GP, with first term, a, common ratio, r,

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

So, the sum of given GP up to 10 terms, with a = √2,

r = √3, n = 10

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{\sqrt{2}((\sqrt{3})^{10} - 1)}{\sqrt{3} - 1}$$

The requires sum, $S_n = \frac{\sqrt{2}((3)^5 - 1)}{\sqrt{3} - 1} = 467.5$

Q. 9. Find the sum of n terms of the series whose r^{th} term is ($r + 2^r$).

Answer : We need to find the sum of n terms of series whose r^{th} term is $r + 2^r$.

$a_r = r + 2^r$

So, n^{th} term, $a_n = n + 2^n$

So, we can find the sum of the series by using summation of the n^{th} term of the given series.

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k + 2^k$$

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k + \sum_{k=1}^{n} 2^k \rightarrow (1)$$

<u>**Note:**</u>

I. Sum of first n natural numbers, 1 + 2 +3+…n,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Second term in (2) is a GP, with first term a = 2, common ratio r = 2.

Sum of n terms of GP, with the first term, a, common ratio, r,

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

So, the sum of given GP, with a = 2, r = 2

$$S_n{}^1 = \frac{a(r^n - 1)}{r - 1} = \frac{2(2^n - 1)}{2 - 1} = 2(2^n - 1)$$

$$S_n = \sum_{k=1}^{n} k + \sum_{k=1}^{n} 2^k$$

The required sum,

$$S_n = \frac{n(n+1)}{2} + 2(2^n - 1)$$

$$S_n = \frac{n^2 + n + 4(2^n) - 4}{2} = \frac{n^2 + n - 4 + (2^{n+2})}{2}$$

The sum of n terms of the series whose r^{th} term is $(r + 2^r)$,

$$S_n = \frac{n^2 + n - 4 + (2^{n+2})}{2}$$